Cut, Analytic Cut and Interpolation 3

Hiroakira Ono (ono@jaist.ac.jp)

School of Proof Society 2024

Birmingham, 9-10 September 2024

Further on Craig's interpolation property

Maehara's method for a given cut-free system consists of showing that every rule R (except cut rule) *admit the transfer of interpolation*, i.e., in every application of the rule R, its lower sequent has always an interpolant, whenever the upper sequent(s) has an interpolant for every partition. Then, how about the analytic cut?

★ Analytic cut is compatible with Maehara's method

(0) To see how analytic cut rule can admit the transfer of interpolation (in many systems), we consider here the case for a single-succedent system **LJ** as an example. Then, analytic cut rule is of the following form, where the formula $\alpha \in \mathit{Sub}(\Gamma, \Delta, \varphi)$.

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \varphi}{\Gamma, \Delta \Rightarrow \varphi}.$$

(i) Take an arbitrary partition $\langle(\Gamma_1, \Delta_1 : \emptyset); (\Gamma_2, \Delta_2 : \varphi)\rangle$ of the lower sequent. We assume first that $\alpha \in \mathit{Sub}(\Gamma_1, \Delta_1)$. Then, take a partition $\langle(\Gamma_2 : \emptyset); (\Gamma_1 : \alpha)\rangle$ of $\Gamma \Rightarrow \alpha$, and also $\langle(\alpha, \Delta_1 : \emptyset); (\Delta_2 : \varphi)\rangle$ of $\alpha, \Delta \Rightarrow \varphi$. By our assumption, there exist an interpolant β of the former and γ of the latter, respectively. Then, we can show that $\beta \to \gamma$ is a required interpolant.

(ii) Next, suppose that $\alpha \in \mathit{Sub}(\Gamma_2, \Delta_2, \varphi)$. We take a partition $\langle(\Gamma_1 : \emptyset); (\Gamma_2 : \alpha)\rangle$ of $\Gamma \Rightarrow \alpha$, and also $\langle(\Delta_1 : \emptyset); (\alpha, \Delta_2 : \varphi)\rangle$ of $\alpha, \Delta \Rightarrow \varphi$. By our assumption, there exist interpolants β' and γ' of each of them. We can show that $\beta' \wedge \gamma'$ is an interpolant for this case.

Thus, from the analytic cut property of **GS5**, it follows that

modal logic **S5** has the Craig's interpolation property.

- Further results on the Craig's interpolation property by Takano

In this way, Craig's interpolation property can be obtained from the analytic cut property in many modal logics, via Maehara's method. Takano proved the Craig's interpolation property of **KB**, **KTB**, **KDB**, **K4B**, **K5** and **K5D**, as a consequence of their (extended) subformula property.

♣ Problems to be solved:

So far so good. But this topic is not explored much and remains still in its early stage. It is quite necessary to develop

1. general, syntactical ways of proving the analytic cut property,
2. semantical or algebraic ways, in particular for substructural logics.

(Is there any good example of a sequent system for a substructural logic with the analytic cut property in which cut elimination fails? There must be.)

As for (1), a recent paper [Ciabattoni-Lang-Ramanayake 23] sheds a new light to syntactic approach to the analytic cut property in a more general setting.

Multiple-succedents systems with restricted rules

Recall that partitions of $\Gamma \Rightarrow \varphi$ of the single-succedent system **LJ** are limited only to those of the form $\langle(\Gamma_1, \emptyset); (\Gamma_2, \Delta)\rangle$, Let us consider a multiple-succedents sequent system, which has rules whose applications are restricted only to single-succedent sequent or even single-antecedent one. The multiple-succedents system $\mathbf{LJ}'$ for intuitionistic logic, which we are going to discuss, and its modal extensions, are such examples. A natural question is:

does Maehara's method work even for such a case?

The rest of my talk will be devoted to some positive results on this question, based on my joint works with T. Kowalski and K. Sano.

Multiple-succedents sequent system $\mathbf{LJ}'$

The system $\mathbf{LJ}'$ is obtained from **LK** by restricting only right implication rule ($\Rightarrow\!\rightarrow$) to the following form. (When we take negation as a primitive symbol, it is necessary to impose the same restriction on the right negation rule.) In the following, we assume that each sequent is of the form $\Gamma \Rightarrow \Delta$, where both Γ and Δ are multisets of formulas.

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\Rightarrow\!\rightarrow)$$

The system $\mathbf{LJ}'$ is introduced by both Maehara (1954) and Umezawa (1955). Cut elimination for $\mathbf{LJ}'$ is mentioned (without proof) in Umezawa (1959),

Sequent system **LBJ** for bi-intuitionistic logic **BiInt**

Bi-intuitionistic logic **BiInt**, which is introduced by C. Rauszer around 1977, is a natural extension of intuitionistic logic by an additional connective, called coimplication $\prec$, which is dual to the usual implication. A sequent system **LBJ** for the logic **BiInt** is obtained from **LJ**$'$ by adding the following rules for coimplication.

Rules for coimplication

$$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \prec \beta \Rightarrow \Delta}\ (\prec\Rightarrow) \qquad \frac{\Gamma \Rightarrow \Lambda, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \alpha \prec \beta, \Lambda, \Pi}\ (\Rightarrow\prec)$$

Rules for implication (just for comparison)

$$\frac{\Gamma \Rightarrow \Lambda, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Delta \Rightarrow \Lambda, \Pi}\ (\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

(Note that $\alpha \prec \beta$ is equal to $\alpha \wedge \neg\beta$ in classical logic.)

♣ How about the cut elimination for **LBJ**?

Cut elimination fails for **LBJ** [Uustalu 04]. Here is a counter-example.

- The sequent $p \Rightarrow q, r \rightarrow ((p \prec q) \wedge r)$ is provable in **LBJ**.

  In fact, both $p \Rightarrow q, p \prec q$ and $p \prec q \Rightarrow r \rightarrow ((p \prec q) \wedge r)$ are provable.
  Thus, by applying cut, which in fact is analytic, we get the required sequent.

- It is not provable in **LBJ** without using cut.

  Suppose otherwise. Consider which rule is applied last in a cut-free proof to get it.
  Only structural rules are possible. It cannot be weakening, as neither
  $p \Rightarrow r \rightarrow ((p \prec q) \wedge r)$ nor $p \Rightarrow q$ is provable. (For the former, consider the case where $p = q = r = 1$.) Also, contraction will make the situation even worse.

There have been several attempts to get cut-free sequent systems for **BiInt**. For instance,

- Nested sequent system [Goré,Postniece,Tiu08],
- Labelled sequent system [Pinto-Uustalu09].

Using the same method for **GS5** we discussed before, we have the following. It implies that **LBJ** is a conservative extension of $\mathbf{LJ}'$.

Theorem

The sequent system **LBJ** *has the subformula property and hence has the analytic cut property.*

Towards the Craig interpolation property for **BiInt**

We will show Craig interpolation for **BiInt** by using Maehara's method. First, we need to discuss what kind of partitions we require.

- Since **LBJ** is a multiple-succedents system, in general we need to consider partitions in the general form $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of a given sequent $\Gamma \Rightarrow \Delta$, where $[\Gamma_1|\Gamma_2] = \Gamma$ and $[\Delta_1|\Delta_2] = \Delta$.

- On the other hand, it will be also necessary to restrict partitions of $\Gamma \Rightarrow \varphi$ only to those of the form $\langle(\Gamma_1, \emptyset); (\Gamma_2, \Delta)\rangle$, because of right implication rule. Considering the duality between implication and *coimplication* in **BiInt**, we will be led to the following definition.

♠ Maehara's method restricted to normal partitions

A partition $\langle(\Gamma_1, \Delta_1); (\Gamma_2, \Delta_2)\rangle$ of $\Gamma \Rightarrow \Delta$ of **LBJ** is normal if it is either of the form $\langle(\Gamma_1, \emptyset); (\Gamma_2, \Delta)\rangle$ or of the form $\langle(\Gamma, \Delta_1); (\emptyset, \Delta_2)\rangle$.

Theorem (CIP of BiInt by Kowalski-Ono17)

Suppose that $\Gamma \Rightarrow \Delta$ *is provable in* **LBJ** *and* $\langle(\Gamma_1, \Delta_1); (\Gamma_2, \Delta_2)\rangle$ *is an arbitrary normal partition of* $\Gamma \Rightarrow \Delta$. *Then there exists an interpolant* φ *for this partition which satisfies that:*

- *both* $\Gamma_1 \Rightarrow \Delta_1, \varphi$ *and* $\varphi, \Gamma_2 \Rightarrow \Delta_2$ *are provable,*
- $Var(\varphi) \subseteq Var(\Gamma_1 \cup \Delta_1) \cap Var(\Gamma_2 \cup \Delta_2)$.

- In the inductive argument of the proof, we can assume the existence of interpolants of the upper sequent(s), only for normal partitions. The proof goes well mostly in the same way as that of **LJ**.

- But, when we show the key fact that analytic cut rule admits the transfer of interpolation for "normal partitions", we will face complications and need a detailed case analysis.

Mints' symmetric method

To show the Craig's interpolation property of the sequent system $\mathbf{LJ}'$, Mints (2001) took a different approach to expanding Maehara's method.

The basic idea is the same as Maehara's method, which consists of inductive construction of interpolants, but accepting arbitrary partitions. As compensation for this, we are forced to allow "*imaginary*" interpolations, in addition to *real* ones.

> *You may compare this with the introduction of "complex numbers" in analysis, by which we can show that every polynomial equation has always solutions.*

- Consider an initial sequent of the form $\alpha \Rightarrow \alpha$ with its partition $\langle(\emptyset : \alpha); (\alpha : \emptyset)\rangle$, for example. We know that $\neg\alpha$ is an interpolant for classical logic, but none can be for intuitionistic logic.

- Now, take the formula α. Clearly, both $\alpha \Rightarrow \alpha$ and $\alpha \Rightarrow \alpha$ are provable. This α looks a bit similar to the original interpolants, though it is placed in the opposite position. Thus, if we are to accept it as another kind of interpolant, we must indicate the positions explicitly where it must be placed.

- So this *imaginary* interpolant would be expressed something as $\alpha\ (p \Rightarrow; \Rightarrow p)$. For the consistency, the usual (Craig) interpolant φ should be expressed as $\varphi\ (\Rightarrow q; q \Rightarrow)$, since it is placed in such positions as indicated in sequents $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Sigma \Rightarrow \Pi$.

- Starting with these two types of interpolants (for initial sequents), interpolants can be constructed inductively in a similar way to Maehara's method. Obviously, in the course of constructions, "interpolants of more complicated form" will appear.

- G. Mints defined interpolants of this kind formally in his paper 2001 and called them multiple interpolants (*M-interpolants*, for short). Here, we will sketch the idea briefly without going into details.

Notation: Let S_1 *and* S_2 *be sequents* $\Sigma_1 \Rightarrow \Pi_1$ *and* $\Sigma_2 \Rightarrow \Pi_2$*, respectively. By the "composition"* $S_1 \circ S_2$ *of* S_1 *and* S_2*), we mean the sequent* $\Sigma_1, \Sigma_2 \Rightarrow \Pi_1, \Pi_2$*. Clearly* $\langle S_1; S_2 \rangle$ *is a partition of the sequent* $S_1 \circ S_2$*.*

Multiple interpolants

Suppose that sequents S_1 and S_2 are given. Let $\varphi_1, \ldots, \varphi_k$ be formulas, and $p_1, \ldots, p_k$ be distinct propositional variables ($0 < k$). Also, $T_1, \ldots, T_m$ be (non-tautological) sequents consisting only of variables from $\{p_1, \ldots, p_k\}$ without any logical connectives ($m \geq 2$).

We say that $(\varphi_1, \ldots, \varphi_k(T_1; \ldots; T_h | T_{h+1}; \ldots; T_m))$ is an multiple interpolant for a partition $\langle S_1; S_2 \rangle$ if the following hold;

1. $Var(\varphi_i) \subseteq Var(S_1) \cap Var(S_2)$ for each $i \leq k$.
2. All sequents $S_1 \circ T_j^*$ for $j \leq h$, and $S_2 \circ T_j^*$ for j with $h < j \leq m$ are provable. Here, T_j^* is the sequent obtained from T_j by replacing each p_i by φ_i.
3. $(T_1; \ldots; T_m)$ is closed. That is, the empty sequent $\Rightarrow$ is derivable from $T_1, \ldots, T_m$ by applying only cut rule and contraction.

Examples of M-interpolants

- A formula φ is a Craig interpolant of a partition $\langle(\Gamma \Rightarrow \Delta); (\Sigma \Rightarrow \Pi)\rangle$ if and only if $(\varphi\ (\Rightarrow q | q \Rightarrow))$ is an M-interpolant of this partition.
- $(\alpha\ (p \Rightarrow \mid \Rightarrow p))$ is an M-interpolant of $\langle(\ \Rightarrow \alpha); (\alpha \Rightarrow\ )\rangle$.
- When $(\varphi, \psi\ (\ \Rightarrow p; p \Rightarrow q \mid q \Rightarrow))$ is an M-interpolant for $\langle(\Gamma \Rightarrow \Delta); (\Sigma \Rightarrow \Pi)\rangle$, all sequents (1) $\Gamma \Rightarrow \Delta, \varphi$, (2) $\varphi, \Gamma \Rightarrow \Delta, \psi$ and (3) $\psi, \Sigma \Rightarrow \Pi$ must be provable.

- The part $(T_1; \ldots; T_m)$ in a given M-interpolant is a list which indicates where each formula in $\varphi_1, \ldots, \varphi_k$ should be placed when it is applied to the partition $\langle S_1; S_2 \rangle$. As a matterr of fact, these lists are constructed only by using two operations, the "composition" and the "union".

If an M-interpolant of the sequent $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ exists for a given partition, then the sequent is provable.

★ Analytic cut is compatible also with Mints' method

Mints applied his method to the multiple-succedents sequent system **LJ**′, for which cut elimination holds. In [O-Sano22], it is shown that the method can be extended also to multiple-succedents sequent systems with the analytic cut property, e.g., **LBJ**. This fact can be assured by showing that the analytic cut can admit the transfer of M-interpolants. As a consequence, we have;

Theorem

Suppose that $\Gamma \Rightarrow \Delta$ *is provable in* **LBJ** *and* $\langle(\Gamma_1, \Delta_1); (\Gamma_2, \Delta_2)\rangle$ *is an arbitrary partition of* $\Gamma \Rightarrow \Delta$*. Then there exists an M-interpolant for this partition.*

♣ But, what does this have to do with Craig interpolation?

♠ Mints' method applied to **LBJ**

There exists a constructive way of getting a Craig interpolant for a partition $\langle S_1; S_2 \rangle$ from a given M-interpolant for the same partition, as long as it is normal.

... just like the real axis in the complex plane ...

Combining this with the previous theorem, we can give an alternative proof of Craig interpolation theorem for **BiInt** [O-Sano22].

- M-interpolants and Craig's interpolants – an example

  - To see how Mints' argument goes, consider the following example.

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

Consider a partition $\langle(\Gamma_1 \Rightarrow \Delta_1); (\Gamma_2 \Rightarrow \Delta_2, \alpha \wedge \beta)\rangle$ of the lower sequent. Corresponding to this, we take partitions $\langle(\Gamma_1 \Rightarrow \Delta_1); (\Gamma_2 \Rightarrow \Delta_2, \alpha)\rangle$ and $\langle(\Gamma_1 \Rightarrow \Delta_1); (\Gamma_2 \Rightarrow \Delta_2, \beta)\rangle$ of left- and right-upper sequents, respectively. By the I.H., there exist M-interpolants. Suppose that they are $(\theta\ (\ \Rightarrow p; p \Rightarrow\ ))$ for the first and $(\varphi\ (q \Rightarrow\ ;\ \Rightarrow q))$ for the second partition, respectively.

- By the procedure of constructing M-interpolants (though we omit the detail), we can get $(\theta, \varphi\ (q \Rightarrow p \mid p \Rightarrow\ ;\ \Rightarrow q))$ as the required M-interpolant of the given partition. Thus, all sequents (1) $\varphi, \Gamma_1 \Rightarrow \Delta_1, \theta$, (2) $\theta, \Gamma_2 \Rightarrow \Delta_2, \alpha \wedge \beta$, and (3) $\Gamma_2 \Rightarrow \Delta_2, \alpha \wedge \beta, \varphi$ are provable.

- Let us suppose in addition that Δ_1 is empty, which makes the partition under consideration normal. Then, from (1) $\Gamma_1 \Rightarrow \varphi \rightarrow \theta$ follows, and from (2) and (3) $\varphi \rightarrow \theta, \Gamma_2 \Rightarrow \Delta, \alpha \wedge \beta$ follows. This means that $\varphi \rightarrow \theta$ is a Craig interpolant of our partition.

Bi-intuitionistic stable tense logic **BiSKt**

- Mints' method seems to be more manageable than the first approach. In [O-Sano22], the above method is used for showing Craig interpolation theorem for other logics, including intuitionistic modal logic $\mathbf{L_4}$ and bi-intuitionistic stable tense logic **BiSKt**.

- **BiSKt** is introduced in [Stell,Schmidt,Rydeheard16] (see also [Sano-Stell17]), which is an extension of **BiInt** with tense operators $\blacklozenge$ and $\Box$, expressing *past possibility* and *future necessity*, respectively. The sequent system **GBiSKt** for **BiSKt** is obtained from **LBJ** by adding the following rules for tense operators.

$$\frac{\blacklozenge\Sigma, \Gamma \Rightarrow \varphi}{\Sigma, \Box\Gamma \Rightarrow \Box\varphi}\ (\Box) \qquad \frac{\varphi \Rightarrow \Delta, \Box\Lambda}{\blacklozenge\varphi \Rightarrow \blacklozenge\Delta, \Lambda}\ (\blacklozenge)$$

- We can show that **GBiSKt** has the analytic cut property. But, this doesn't imply that it has the SP, since neither of ($\Box$) and ($\blacklozenge$) are analytic.

- Let **GBiSKt**^{s} be **GBiSKt** restricting applications of rules ($\Box$) and ($\blacklozenge$) under the conditions $\blacklozenge\Sigma \subseteq \mathit{Sub}(\Gamma, \varphi)$ and $\Box\Lambda \subseteq \mathit{Sub}(\varphi, \Delta)$, respectively.

$$\frac{\blacklozenge\Sigma, \Gamma \Rightarrow \varphi}{\Sigma, \Box\Gamma \Rightarrow \Box\varphi}\ (\Box) \qquad \frac{\varphi \Rightarrow \Delta, \Box\Lambda}{\blacklozenge\varphi \Rightarrow \blacklozenge\Delta, \Lambda}\ (\blacklozenge)$$

  Then, it is also sequent system for **BiSKt** with the SP.

- On the other hand, it is *not necessary to take* **GBiSKt**^{s} and is enough to take the system **GBiSKt** to get the Craig's interpolation theorem, since interpolants are concerned only with variables appearing in common (as remarked in [Takano92]).

♣ Analytic cut property and nested sequent systems

We remark here that essentially the same idea as M-interpolants is used in the paper [Lyon,Tiu,Goré,Clouston 2020]. In the paper, an alternative proof of Craig interpolation property for **BiInt** is given by using a cut-free nested sequent system in [Goré,Postniece,Tiu08]. Mints' "closure condition" is called the "orthogonality condition" there.

Are there any connections, in general?

Semantical approach to Craig's interpolation property

♣ How about semantical approach to the Craig's interpolation property of these logics?

This is an ongoing work with Sano. By extending existing semantical techniques, i.e., bisimulation products, and also super-amalgamation property in parallel, we have already obtained some general results on the Craig's interpolation property (and even the Lyndon interpolation property) of extensions of both **BiInt** and **BiSKt**.