# Cut, Analytic Cut and Interpolation 2

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# Beyond the scope of cut elimination

So far, we have seen that;

- Cut elimination is undoubtedly one of the most important results in proof theory.
- As we have seen, many important logical properties can be obtained from cut elimination via subformula property.

But, cut elimination holds for rather limited number of sequent systems. For instance, it is known that for modal logic **S5** it is hard to give a simple, cut-free sequent system.

- The sequent system **GS5** for modal logic **S5**

The system **GS5** is obtained from **LK** by adding the following rules $(\Box \Rightarrow)$ and $(\Rightarrow \Box 2)$ for $\Box$. Take note that if we restrict $\Delta$ in $(\Rightarrow \Box 2)$ to empty set, we get the (cut-free) system **GS4**.

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow \Box\Delta, \alpha}{\Box\Gamma \Rightarrow \Box\Delta, \Box\alpha}\ (\Rightarrow \Box 2)$$

## Cut elimination fails for **GS5** [Ohnishi-Matsumoto57]

- For instance, take $p \Rightarrow \Box\neg\Box\neg p$, which is an instance of the modal axiom B. It is provable in **GS5**, but cannot be provable without using cut rule in it. This will be seen by examining the following proof of $p \Rightarrow \Box\neg\Box\neg p$ with cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box\neg p \Rightarrow \Box\neg p}{\Rightarrow \neg\Box\neg p, \Box\neg p}}{\Rightarrow \Box\neg\Box\neg p, \Box\neg p}\,(\Rightarrow\Box 2) \quad \dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{\Box\neg p, p \Rightarrow}\,(\Box \Rightarrow)}{p \Rightarrow \Box\neg\Box\neg p}\,(cut)$$

# Analytic cut

An application of cut rule in $P$ is analytic if it is of the following form;

$$\frac{\Theta \Rightarrow \Sigma, \alpha \quad \alpha, \Pi \Rightarrow \Lambda}{\Theta, \Pi \Rightarrow \Sigma, \Lambda}\ (\text{analytic cut}).$$

where $\alpha$ belongs to $Sub(\Theta, \Pi, \Sigma, \Lambda)$, the set of all subformulas of formulas in $\Theta, \Pi, \Sigma, \Lambda$.

- A given proof $P$ of a sequent $\Gamma \Rightarrow \alpha$ has the analytic cut property, if every application of cut rule in $P$ is analytic
- A sequent system **S** has the analytic cut property, if every sequent provable in **S** has a proof with the analytic cut property.

Obviously, the following holds.

In a given sequent system **S**, suppose that every rule except cut rule is analytic. If **S** has the analytic cut property (ACP) then **S** has the subformula property (SP).

Analytic cut is discussed in the paper "Analytic cut" by R.M. Smullyan (1968) in which he mentioned that;

> *The real importance of cut-free proofs is not the elimination of cuts per se, but rather that such proofs obey the subformula principle.*

- Then, does the SP imply the ACP?

## Analytic cut property as local subformula property

- The ACP can be regarded as a *local subformula property*. So, our question is whether the global SP implies the local one, or not.

- Consider the following example in a system **S** with the subformula property. Suppose that the whole proof $P$ in the following has the subformula property, and also that $\alpha \wedge \beta \notin \mathit{Sub}(\Gamma, \Sigma, \varphi)$.

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \wedge \beta \quad \alpha \wedge \beta, \alpha, \Sigma \Rightarrow \varphi}{\alpha, \Gamma, \Sigma \Rightarrow \varphi}\,(\mathit{cut})}{\alpha \wedge \beta, \Gamma, \Sigma \Rightarrow \varphi}$$

Since the above application of cut rule is not analytic, the proof $P$ itself doesn't have the ACP.

- Subformula property vs analytic cut property [Kowalski-O 17]

Suppose that every rule except cut rule is analytic in a sequent system **S**. Then, **S** has the subformula property if and only if it has the analytic cut property.

It's enough to show that every proof with the subformula property of a sequent $\Gamma \Rightarrow \Delta$ can be transformed into a proof of the same sequent with the analytic cut property.

1. Let $P$ be a proof of $\Gamma \Rightarrow \Delta$ with the SP. We will prove our result by using the induction on the number $|Sub(\Gamma, \Delta)|$ of formulas in the set $Sub(\Gamma, \Delta)$. It suffices to consider the case that there exists at least one application of non-analytic cut rule in $P$.

2. Take one of such an application of non-analytic cut which is (one of) the lowest in $P$ among them, whose cut formula is $\theta$ and lower sequent is $\Sigma \Rightarrow \Pi$. As this cut is non-analytic, $\theta \notin Sub(\Sigma, \Pi)$. Let $Q$ be the subproof up to $\Sigma \Rightarrow \Pi$ in $P$.

$$\dfrac{\dfrac{\Sigma_1 \Rightarrow \Pi_1, \theta \quad \theta, \Sigma_2 \Rightarrow \Pi_2}{\Sigma \Rightarrow \Pi}\,(cut)}{\vdots}$$
$$\Gamma \Rightarrow \Delta$$

3. As $P$ has the SP by our assumption, $Sub(\Sigma, \Pi) \subseteq Sub(\Gamma, \Delta)$ holds, while $\theta \in Sub(\Gamma, \Delta) \backslash Sub(\Sigma, \Pi)$. Hence, $|Sub(\Sigma, \Pi)| < |Sub(\Gamma, \Delta)|$. Now,

- there must be $Q^*$ of $\Sigma \Rightarrow \Pi$ with the SP (by the SP of the system **S**),
- there exists $Q^{**}$ of $\Sigma \Rightarrow \Pi$ with the ACP (by I.H.),
- the proof $P'$ of $\Gamma \Rightarrow \Delta$, obtained from $P$ by replacing $Q$ by $Q^{**}$ has the SP. The number of applications of non-analytic cut in $P'$ is less than that in $P$.

4. By repeating this procedure, all applications of non-analytic cut are eventually removed. Consequently, we get a proof of $\Gamma \Rightarrow \Delta$ with the analytic cut property. (Thus, in fact, we use the double induction.)

- Does **GS5** have the analytic cut property?
  - Let us examine again the previous proof of $p \Rightarrow \Box\neg\Box\neg p$ in **GS5** with cut.

$$\dfrac{\dfrac{\dfrac{\Box\neg p \Rightarrow \Box\neg p}{\Rightarrow \neg\Box\neg p, \Box\neg p}}{\Rightarrow \Box\neg\Box\neg p, \Box\neg p}\,(\Rightarrow \Box 2) \qquad \dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{\Box\neg p, p \Rightarrow}\,(\Box \Rightarrow)}{p \Rightarrow \Box\neg\Box\neg p}\,(cut)$$

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  - Then, you will see that this application of cut rule is analytic.

$$\dfrac{\dfrac{\dfrac{\Box\neg p \Rightarrow \Box\neg p}{\Rightarrow \neg\Box\neg p, \Box\neg p}}{\Rightarrow \Box\neg\Box\neg p, \Box\neg p} \quad \dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{\Box\neg p, p \Rightarrow}}{p \Rightarrow \Box\neg\Box\neg p}\,(a.cut)$$

## Theorem (Takano92)

*The sequent system* **GS5** *has the analytic cut property.*

1. M. Takano, Subformula property as a substitute for cut-elimination in modal propositional logics, Mathematica Japonica 37, (1992) pp.1192-1145.
2. M. Takano, A modified subformula property for the modal logics K5 and K5D, Bulletin of the Section of Logic 30, (2001) pp.115-122.

Takano gave a syntactic way of proving the analytic cut property in (1), and about 10 years later he published a short note on a simple, semantic way of proving the subformula property in (2).

# Syntactic approach to analytic cut property

1. The proof goes similarly to the proof of cut elimination, which consists of considering a proof of a given sequent $\Gamma \Rightarrow \Delta$ containing a single application of non-analytic cut as the last inference, and finding a proof of the same sequent with the analytic cut property.

2. But a certain difficulty occurs in the following case:

$$\dfrac{\dfrac{\Box\Gamma \Rightarrow \Box\Delta, \Box\alpha, \beta}{\Box\Gamma \Rightarrow \Box\Delta, \Box\alpha, \Box\beta}\,(\Rightarrow \Box 2) \quad \dfrac{\alpha, \Sigma \Rightarrow \Pi}{\Box\alpha, \Sigma \Rightarrow \Pi}\,(\Box \Rightarrow)}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Box\beta, \Pi}\,(cut)$$

Here both of upper sequents are results of the right and left $\Box$ rules, respectively, and the left principal formula $\Box\beta$ is not the cut formula $\Box\alpha$, while the right one is this $\Box\alpha$. How to reduce this cut to a simpler one?

3. To resolve this, Takano showed the following nontrivial fact.

*If a sequent* $\Box\Theta \Rightarrow \Box\Lambda_1, \Box\varphi, \Box\Lambda_2$ *has a proof with the analytic cut property whose last inference is* $(\Rightarrow \Box 2)$*, then the sequent* $\Box\Theta \Rightarrow \Box\Lambda_1, \varphi, \Box\Lambda_2$ *has a proof with the analytic cut property. (Note that the principal formula of the inference* $(\Rightarrow \Box 2)$ *is not necessary* $\Box\varphi$*.)*

By taking $\alpha$ for $\varphi$ in this lemma, we can reduce the previous proof to the following with a smaller grade.

$$\frac{\Box\Gamma \Rightarrow \Box\Delta, \alpha, \Box\beta \quad \alpha, \Sigma \Rightarrow \Pi}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Box\beta, \Pi}\ (cut)$$

# Semantical proof of subformula property

A semantical proof of the subformula property of **GS5**, due to Takano, goes in parallel with the semantical proof of cut elimination of **GS4**. We will show that $\Gamma \Rightarrow \Delta$ is false in an **S5**-frame, by assuming that the sequent $\Gamma \Rightarrow \Delta$ does not have any proof with the SP.

1. **GS5**-consistency relative to $\Omega$

   For a given sequent $\Gamma \Rightarrow \Delta$, let $\Omega = \mathit{Sub}(\Gamma, \Delta)$. (Thus, $\Omega$ is closed under subformulas.) For any pair $(\Psi, \Pi)$ of subsets $\Psi$ and $\Pi$ of $\Omega$, a proof $P$ of the sequent $\Psi \Rightarrow \Pi$ (if there exists) is an $\Omega$-proof in **GS5** if every formula appearing in $P$ belongs to $\Omega$. Otherwise, the pair $(\Psi, \Pi)$ is said to be **GS5**-consistent relative to $\Omega$ ($\Omega$-consistent, for short).

   By our assumption, $(\Gamma, \Delta)$ is an example of $\Omega$-consistent pairs.

## 2. Extension lemma

[Analytic extension] If $(\Sigma, \Theta)$ is $\Omega$-consistent, then for any formula $\gamma$ in $\Omega$ either $(\Sigma \cup \{\gamma\}, \Theta)$ or $(\Sigma, \Theta \cup \{\gamma\})$ is $\Omega$-consistent.

Proof. In fact, if each of $\Sigma, \gamma \Rightarrow \Theta$ and $\Sigma \Rightarrow \Theta, \gamma$ has an $\Omega$-proof, then applying the cut rule with the cut formula $\gamma$ ($\in \Omega$) to them, we get an $\Omega$-proof of $\Sigma \Rightarrow \Theta$. This is a contradiction

Then, we have the following.

[Extension lemma] If $(\Sigma_0, \Theta_0)$ is $\Omega$-consistent, then there exists a maximal $\Omega$-consistent extension $(\Sigma, \Theta)$ of $(\Sigma_0, \Theta_0)$ satisfying that $\Sigma \cup \Theta = \Omega$.

## 3. Constructing an **S5**-frame

Define a model $\langle W, S, V\rangle$ as follows.

- $W$ is the set of all subsets $\Sigma$ of $\Omega$ such that $(\Sigma, \Sigma^c)$ is a maximal $\Omega$-consistent pair such that (1) $\Gamma \subseteq \Sigma$ and (2) $\Delta \subseteq \Sigma^c$, where $\Sigma^c = (\Omega \backslash \Sigma)$,
- For every $\Sigma, \Lambda \in W$, the binary relation $S$ on $W$ is defined by: $\Sigma\ S\ \Lambda$ iff $\Sigma_\Box = \Lambda_\Box$. Recall that $\Phi_\Box = \{\gamma : \Box\gamma \in \Phi\}$.
- A valuation $V$ on $\langle W, S\rangle$ is defined by $V(p) = \{\Sigma \in W : p \in \Sigma\}$ for each propositional variable $p \in \Omega$.

Obviously, $\langle W, S\rangle$ is a (finite) **S5** frame, i.e., $S$ is reflexive, transitive and symmetric. Using that $\Sigma \cup \Theta = \Omega$ holds for every maximal $\Omega$-consistent pair $(\Sigma, \Theta)$, we have that $V(\alpha) = \{\Sigma \in W; \alpha \in \Sigma\}$ for every formula $\alpha \in \Omega$.

## 4. Subformula property of **GS5**

[model existence lemma] If a pair $(\Gamma, \Delta)$ is $\Omega$-consistent with $\Omega = Sub(\Gamma, \Delta)$, then there exists a **S5**-model $\langle W, S, V \rangle$ such that every $\alpha \in \Gamma$ is true and every $\beta \in \Delta$ is false.

By taking the contraposition, we have the following subformula property, and hence the analytic cut property of **GS5**.

### Theorem

*If a sequent $\Gamma \Rightarrow \Delta$ is provable in* **GS5**, *there exists a proof $P$ of $\Gamma \Rightarrow \Delta$ such that every formula appearing in $P$ is a subformula of a formula either in $\Gamma$ or in $\Delta$.*

## ♣ Some remarks

- Similarly, Takano gives both syntactical and semantical proofs of the subformula property of sequent systems for modal logics **KB**, **KTB**, **KDB** and **K4B**, in [Takano92, Takano01], respectively.
- Moreover, he extended the notion of the subformula property, called the extended subformula property, and showed that the extended subformula property holds for sequent systems for **K5** and **K5D**, in [Takano01, Takano18].
- Related topics are discussed also in several papers by [Lahav-Avron] and [Lahav-Zohar].

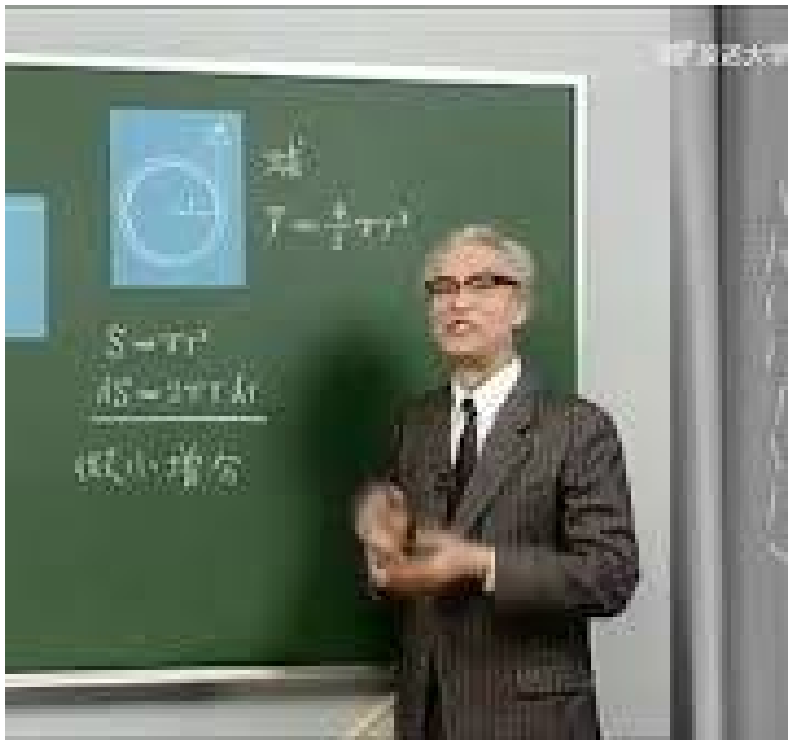
# Maehara's method for showing the CIP

My main concern is now to see how broadly the analytic cut property can be applied in deriving the Craig interpolation property. For that purpose, we will explain briefly Maehara's method in the following. Recall that;

- A logic **L** has the Craig's interpolation property (CIP), if for all formulas $\alpha$ and $\beta$ such that $\alpha \to \beta$ is provable in **L**, there exists a formula $\varphi$, called an interpolant of $\alpha \to \beta$, such that

  (1) both $\alpha \to \varphi$ and $\varphi \to \beta$ are provable in **L**,

  (2) $Var(\varphi) \subseteq Var(\alpha) \cap Var(\beta)$.



Prof. Shôji Maehara (1927-1992)

## Maehara's method

W. Craig proved the CIP of classical logic in a model-theoretic way. In 1960, S. Maehara succeeded to give a syntactic proof of the CIP as a consequence of cut elimination. We will first give a brief sketch of his proof for the case of **LK** (with constants 0 and 1).

- For our convenience, sequents are regarded as expressions of the form $\Gamma \Rightarrow \Delta$ with multisets $\Gamma$ and $\Delta$ of formulas. A partition of a multiset $\Gamma$ is a pair $[\Gamma_1|\Gamma_2]$ of multisets $\Gamma_1$ and $\Gamma_2$ whose *multiset union* is $\Gamma$.
- A partition of a sequent $\Gamma \Rightarrow \Delta$ is an ordered pair $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$, where $[\Gamma_1|\Gamma_2]$ and $[\Delta_1|\Delta_2]$ are partitions of multisets $\Gamma$ and $\Delta$, respectively. For example, $\langle(\alpha, \beta : \gamma); (\gamma, \beta : \delta)\rangle$ is a partition of a sequent $\alpha, \gamma, \beta, \beta \Rightarrow \delta, \gamma$. This partition can be sometimes expressed in a more intuitive way as $\langle(\Gamma_1 \Rightarrow \Delta_1); (\Gamma_2 \Rightarrow \Delta_2)\rangle$.

Using Maehara's method, Craig's interpolation theorem of **LK** can be shown in a more general form.

## Theorem

*Suppose that a sequent* $\Gamma \Rightarrow \Delta$ *is provable in* **LK**. *Then, for any partition* $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ *of* $\Gamma \Rightarrow \Delta$, *there exists a formula* $\varphi$, *called an interpolant for this partition, which satisfies*

(1) *both sequents* $\Gamma_1 \Rightarrow \Delta_1, \varphi$ *and* $\varphi, \Gamma_2 \Rightarrow \Delta_2$ *are provable in* **LK**,

(2) $Var(\varphi) \subseteq Var(\Gamma_1, \Delta_1) \cap Var(\Gamma_2, \Delta_2)$.

- The proof of the above theorem is outlined as follows.

  (i) Take any cut-free proof $P$ of $\Gamma \Rightarrow \Delta$.

  (ii) For each sequent $\Pi \Rightarrow \Lambda$ in $P$ and each of its partitions, give an interpolant one after another from the top of $P$, by using the induction on the length of its proof (i.e., the subproof of $\Pi \Rightarrow \Lambda$ in $P$).

- Obviously, interpolants obtained in this way depends on the choice of the cut-free proof $P$.

- In the following, $\emptyset$ denotes the empty multiset, and $\alpha, \Gamma$ denotes the multiset consisting of a formula $\alpha$ together with formulas in a multiset $\Gamma$.

Let us glance at how the proof goes, by considering just a few case.

- The case where $\Pi \Rightarrow \Lambda$ is an initial sequent $\alpha \Rightarrow \alpha$.
  - There are four partitions.
    (a) $\langle(\alpha : \alpha); (\emptyset : \emptyset)\rangle$,
    (b) $\langle(\alpha : \emptyset); (\emptyset : \alpha)\rangle$,
    (c) $\langle(\emptyset : \alpha); (\alpha : \emptyset)\rangle$,
    (d) $\langle(\emptyset : \emptyset); (\alpha : \alpha)\rangle$.
  - For each partition, an interpolant is (a) 0, (b) $\alpha$, (c) $\neg\alpha$, and (d) 1, respectively.
  - In the partition (c), $\neg\alpha$ is an interpolant as both $\Rightarrow \alpha, \neg\alpha$ and $\neg\alpha, \alpha \Rightarrow$ are provable. But we note that the former sequent, which is essentially equivalent to $\Rightarrow \alpha \vee \neg\alpha$ in **LK**, cannot be provable in general in **LJ**.

- The case where $\Pi \Rightarrow \Lambda$ is the lower sequent of the $(\Rightarrow\rightarrow)$-rule.

$$\frac{\alpha, \Sigma \Rightarrow \Theta, \beta}{\Sigma \Rightarrow \Theta, \alpha \rightarrow \beta}\ (\Rightarrow\rightarrow)$$

- By the hypothesis of induction, we assume that $\alpha, \Sigma \Rightarrow \Theta, \beta$ has an interpolant for each partition. Here we consider only the case where a partition $\langle(\Sigma_1 : \Theta_1); (\Sigma_2 : \Theta_2, \alpha \rightarrow \beta)\rangle$ of the lower sequent $\Sigma \Rightarrow \Theta, \alpha \rightarrow \beta$. Take a partition $\langle(\Sigma_1 : \Theta_1); (\alpha, \Sigma_2 : \Theta_2, \beta)\rangle$ of $\alpha, \Sigma \Rightarrow \Theta, \beta$. By our assumption, there is an interporant $\gamma$ for this partition such that both sequents $\Sigma_1 \Rightarrow \Theta_1, \gamma$ and $\gamma, \alpha, \Sigma_2 \Rightarrow \Theta_2, \beta$ are provable.

  From the latter sequent, $\gamma, \Sigma_2 \Rightarrow \Theta_2, \alpha \rightarrow \beta$ can be derived. Thus, $\gamma$ is also an interpolant of $\Sigma \Rightarrow \Theta, \alpha \rightarrow \beta$ for the present partition, as it is easily seen that this $\gamma$ satisfies also the required condition on variables.

## ♣ Partitions in single-succedent sequent systems

- A slight modification of the definition of partitions of sequents will be necessary in single-succeedent sequent systems like **LJ**. In this case, for a given sequent $\Gamma \Rightarrow \delta$, we consider only partitions of the form $\langle(\Gamma_1 : \emptyset); (\Gamma_2 : \delta)\rangle$, where $[\Gamma_1|\Gamma_2]$ is a partition of $\Gamma$. In this way, we can avoid an inconvenience which is caused by the partition of the form $\langle(\emptyset : \alpha); (\alpha : \emptyset)\rangle$ mentioned before.

- Also for a single-succedent sequent system for basic substructural logics which lacks exchange rule, like **FL** or $\mathbf{FL_w}$, it is necessary to define a partition of a sequent $\Gamma \Rightarrow \delta$ as a triple $[\Gamma_1|\Gamma_2|\Gamma_3]$ such that $\Gamma_1, \Gamma_2, \Gamma_3$ is exactly same as $\Gamma$ as *ordered sequences* of formulas. This is due to the presence of left- and right-implications. In this case, we say that $\varphi$ is an interpolant of $\Gamma \Rightarrow \delta$ for the partition $[\Gamma_1|\Gamma_2|\Gamma_3]$, when (1) both $\Gamma_2 \Rightarrow \varphi$ and $\Gamma_1, \varphi, \Gamma_3 \Rightarrow \delta$ are provable, and (2) $Var(\varphi) \subseteq Var(\Gamma_2) \cap Var(\Gamma_1, \Gamma_3)$.