Cut, Analytic Cut and Interpolation 1

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Cut elimination and its consequences

♣ *Cut elimination is one of most significant tools in proof-theoretic approach, as many useful logical results are obtained from cut elimination through detailed analysis of structures of cut-free proofs.*

♣ *Searching for possibilities of widening the scope of cut elimination. I will focus my attention on the subformula property and the analytic cut property, and their applications.*

♣ *In my 3rd lecture, I will explain how the Craig's interpolation property of bi-intuitionistic (tense) logic can be obtained from the analytic cut property, hoping that it will give a promising example of our attempts.*

Sequent systems

Throughout our talk, we will concentrate on Gentzen's formulation of sequent systems, and will not touch on any of nested, labelled and hyper-sequent systems.

First, let us take **LJ** for intuitionistic logic as a basic system. So, a (single-succedent) sequent is defined to be an expression of the following form for some $m \geq 0$, where $\alpha_1, \ldots, \alpha_m, \beta$ are (possibly empty) formulas;

$$\alpha_1, \ldots, \alpha_m \Rightarrow \beta$$

Discussions in the following can be applied also to sequent systems for multiple- succedents sequents, like **LK** for classical logic.

The system **LJ** (with constants 0 and 1) consists of initial sequents and the following three kinds of rules.

- Initial sequents
  - $\alpha \Rightarrow \alpha$
  - $0 \Rightarrow$
  - $\Rightarrow 1$

- Rules
  - Rules for logical connectives
  - Cut
  - Structural rules

- Rules for disjunction, conjunction and implication

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \varphi}{\Sigma, \Gamma, \alpha \to \beta, \Delta \Rightarrow \varphi} \ (\to \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\Rightarrow \to)$$

The negation $\neg\alpha$ of a formula α is defined by $\neg\alpha = \alpha \to 0$.

- Cut

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Delta \Rightarrow \varphi}{\Sigma, \Gamma, \Delta \Rightarrow \varphi}\ (\mathrm{cut})$$

- Structural rules

(e) exchange rule: $\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$

(c) contraction rule: $\dfrac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(i) left-weakening rule: $\dfrac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(o) right-weakening rule: $\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$

(w) weakening rules: (i) + (o)

Cut elimination

Cut elimination for a sequent system **S** says:

- if a sequent is provable in **S** then it is also provable in **S** without any application of cut rule.

G. Gentzen showed that

Cut elimination holds for LK and LJ.

Subformula property

- A given proof P of a sequent $\Gamma \Rightarrow \alpha$ is said to have the subformula property, if P contains only the subformulas of the formulas in $\Gamma \Rightarrow \alpha$.

- A sequent system **S** is said to have the subformula property, if every sequent provable in **S** always has a proof with the subformula property.

- analytic rules
  - A rule R is analytic, when every formula in *upper sequents* of R appears also as a subformula of its consequence (i.e., its *lower sequent*).

Every rule R of **LJ** (and also of **LK**) except cut rule is analytic. Hence, if a proof is cut-free, i.e., doesn't contain any application of cut rule, then it has the subformula property. Thus, from cut elimination we have immediately the following.

Theorem

Both **LJ** *and* **LK** *have the subformula property.*

Consequences of cut elimination for **LJ**

Many logical properties of a given logic **L** can be obtained from cut elimination of the sequent system **S** for **L**. In the following we often identify a logic **L** with the corresponding system **S**, as long as no confusions may occur.

Decidability of **LJ**

To check whether a given sequent is provable or not, it is enough to search for a proof of this sequent among those with the subformula property. Gentzen showed that this searching procedure can be carried out in finitely many steps for the case of **LJ** and **LK**, with the help of weakening and contraction rules. .

- Disjunction property of **LJ**

LJ has the *disjunction property*, i.e., for all formulas α and β, if $\alpha \vee \beta$ is provable in **LJ** then either α or β is provable in it.

This can be shown by checking all possibilities of the last application of a rule in a given cut-free proof of $\alpha \vee \beta$. It is neither contraction nor weakening, and it must be $(\Rightarrow \vee)$. On the other hand, the same argument fails for **LK**. Consider e.g., the standard cut-free proof of a formula $p \vee \neg p$.

- Craig's interpolation property (CIP) of **LJ**

LJ has the *Craig's interpolation property* (CIP). That is, if for all formulas α and β such that $\alpha \to \beta$ is provable in **LJ**, then there exists a formula φ, called an *interpolant* of $\alpha \to \beta$, such that

- both $\alpha \to \varphi$ and $\varphi \to \beta$ are provable in **LJ**,
- $\mathit{Var}(\varphi) \subseteq \mathit{Var}(\alpha) \cap \mathit{Var}(\beta)$.

A well-known syntactic proof is given by S. Maehara (1960). It consists of an explicit construction of such an interpolant in an inductive way, once a cut-free proof of $\alpha \to \beta$ is given.

Substructural aspects

- Are these results affected by the presence or the lack of structural rules?

To see this, we introduce sequent systems of basic substructural logics. Roughly speaking, they are obtained from **LJ** by deleting some or all of structural rules. Our basic system **FL**, called *full Lambek calculus*, is the sequent system obtained from **LJ** by deleting all structural rules, but with small adjustments for constants, shown below.

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0\mathrm{w}) \qquad \frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi}\ (1\mathrm{w})$$

- When either contraction or (left) weakening is missing, a comma in the antecedent of a sequent may not always denote a standard conjunction. To amend this, sometimes we introduce a new logical connective $\cdot$, called the fusion or the multiplicative conjunction, to express commas explicitly in general setting.

  - Rules for fusion

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

- When exchange rule is missing, it is natural to introduce two kinds of implication, the left-implication and the right-implication. But, here we will not go into the details to avoid further complications.

Systems for basic substructural logics

- $\mathbf{FL_e} = \mathbf{FL} + (e)$, $\mathbf{FL_w} = \mathbf{FL} + (w)$, $\mathbf{FL_c} = \mathbf{FL} + (c)$
- $\mathbf{FL_{ew}} = \mathbf{FL} + (e) + (w)$, $\mathbf{FL_{ec}} = \mathbf{FL} + (e) + (c)$

- In the same way, we can introduce *multiple- succedents* sequent systems, by taking **LK** instead of **LJ**, and then deleting some or all of structural rules. (Note that we need to consider here also right-structural rules.)

- For example, the system for *multicative-additive linear logic* (MALL) is obtained from **LK** by deleting both weakening rules and contraction rules. Extensions of this logic are said to be involutive and are sometimes expressed by prefixing **In**, e.g. $\mathbf{InFL_e}$ (= MALL), $\mathbf{InFL_{ew}}$ and $\mathbf{InFL_{ec}}$. We omit further details (see my joint book [GJKO07]).

♦ Many well-known non-classical logics can be regarded as substructural logics. For example,

- Lambek calculus for categorial grammer is a fragment of **FL**,
- Girard's linear logic MALL,
- relevant logics are extensions of $\mathbf{FL_{ec}}$ (often) with the distributive law,
- Johansson's minimal logic is $\mathbf{FL_{ec}}$ with left-weakening rule,
- fuzzy logics and Łukasiewicz's many-valued logics are extensions of $\mathbf{FL_{ew}}$.

Cut elimination for basic substructural logics

Cut elimination can be proved for most of basic substructural logics, by using a slight modification (in fact, a simplification) of standard Gentzen's argument.

1. The following two procedures are used for a given application J of cut rule in a proof. That is,

   (i) to replace J by another application of cut whose cut formula is *simpler* than that of J ... measured by grade = the length of the cut formula,

   (ii) to replace J by another whose proof is *shorter* than that of J ... measured by height = the total number of (occurrences) of sequents over the cut rule under consideration.

2. For basic substructural logics without contraction rules, it is possible to eliminate each application of cut rule by using the double induction on grade and height.

3. For each substructural logic which lacks only weakening rules, the following rule, called the extended cut rule (e-cut), is used as a substitute of mix rule of Gentzen.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \varphi}{\Gamma, \Delta_\alpha \Rightarrow \varphi}\ (\text{e-cut})$$

Here, Δ_α denotes a sequence of formulas obtained from Δ by deleting some (at least one and not necesssarily all) occurrences of α.

Notice that cut and e-cut are replaceable with each other, because of contraction and exchange. Then by using also the double induction on grade and height, we can show the e-cut elimination, and hence the cut elimination. In conclusion, we have the following.

Theorem

Cut elimination holds for $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, $\mathbf{InFL_e}$, $\mathbf{InFL_{ew}}$ *and* $\mathbf{InFL_{ec}}$.

On the other hand, cut elimination fails for $\mathbf{FL_c}$ (if without fusion) as shown in [Bayu Surarso-O96].

♣ It should be remarked that the present argument using e-cut with the double induction of the grade and the height, instead of Gentzen' using mix rule with the grade and the rank, work wells even for **LJ** and **LK**.

This idea is originally introduced in [O-Komori85]. See [O19] for the details.

Consequences of cut elimination in substructural logics

- Many logical properties of substructural logics, like subformula property and Craig's interpolation property, follow from cut elimination, just like the case of **LJ**. (See e.g. [O98] for details.)

- Disjunction property holds as long as a given cut-free system lacks the right-contraction rule.

  Disjunction property holds for $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, $\mathbf{InFL_e}$ *and* $\mathbf{InFL_{ew}}$, *while it fails for* $\mathbf{InFL_{ec}}$.

♣ Decidability results are strongly affected by contraction rules.

- Observe that in any cut-free proof, the upper sequent(s) is simpler than its lower sequent. in each rule except contraction rule(s). Thus, for a cut-free system without contraction rule(s), it is easily seen with the help of the subformula property that every proof-search will terminate in a finite number of steps.

  We note here that the decidability of the quantified logic without contraction rules (the quantified $\mathbf{InFL_{ew}}$) is already mentioned by Hao Wang in his book [Wang63].

  $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{InFL_e}$ *and* $\mathbf{InFL_{ew}}$ *are decidable.*

- On the other hand, in $\mathbf{FL_{ec}}$ and $\mathbf{InFL_{ec}}$, as contraction rules (but without weakening rules) may produce possibly distinct sequents one after another in the proof-searching, it is not so clear whether it (even if restricted to *irredandunt* ones) will eventually terminate.

- To guarantee the finiteness of the number, S. Kripke in 1959 introduced an ingenious idea in his *sketch* of a proof of the decidability of the *implicational fragment* of $\mathbf{FL_{ec}}$. He employed the following, purely combinatorial lemma.

  [Kripke's lemma] For each positive integer m, the direct product $\langle \mathbb{N}^m, \leq \rangle$ doesn't contain any infinite antichain, where $\mathbb{N}$ is the set of natural numbers and $\leq$ is the product order.

Combining it with König's Lemma, the above result is extended to the decidability of $\mathbf{FL_{ec}}$ and $\mathbf{InFL_{ec}}$ ([Meyer66], [Dunn86] and [Kiriyama-O91]). Thus,

Theorem

Any of $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, $\mathbf{InFL_{ew}}$ *and* $\mathbf{InFL_{ec}}$ *is decidable. Also, any of quantified logics* $\mathbf{QFL}$, $\mathbf{QFL_e}$, $\mathbf{QFL_w}$ *and* $\mathbf{QFL_{ew}}$ *(even with function symbols) is decidable.*

In contrast, the following undecidability results are known.

Theorem

i. $\mathbf{FL_c}$ *is undecidable [Chvalovský-Horčík16].*

ii. *The (positive fragment of) relevant logic* $\mathbf{R}$*, which is obtained from* $\mathbf{InFL_{ec}}$ *by adding the distributive law, is undecidable [Urquhart84].*

iii. *Quantified* $\mathbf{QFL_{ec}}$ *is undecidable [Kiriyama-O91].*

Semantical approach to cut elimination

Here, we give an example of semantical proofs of cut elimination using Kripke models, for the standard sequent system **GS4** for modal logic **S4**.

GS4 is obtained from **LK** by adding rules $(\Box \Rightarrow)$ and $(\Rightarrow \Box 1)$ for $\Box$. Note that syntactic proofs of cut elimination for **GS4** are given in [Curry52, Ohnishi-Matsumoto57].

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Rightarrow \Box 1)$$

• Hereafter Greek capital letters, like Γ and Δ, denote finite sets, not multisets, of formulas.

1. $\mathbf{GS4}^{-}$-consistency

Let $\mathbf{GS4}^{-}$ be the system **GS4** without cut rule, and $\Gamma \Rightarrow \Delta$ be an arbitrary sequent which is not provable in $\mathbf{GS4}^{-}$. We show that $\Gamma \Rightarrow \Delta$ is not provable in **GS4**, or alternatively, $\Gamma \Rightarrow \Delta$ is false in an **S4**-model. We take Ω for $Sub(\Gamma, \Delta)$, the set of all subformulas of some formulas in $\Gamma \Rightarrow \Delta$. Obviously, Ω is finite.

- A pair (Σ, Θ) of subsets Σ and Θ of Ω is $\mathbf{GS4}^{-}$-consistent (for short, $\star$-consistent) if the sequent $\Sigma \Rightarrow \Theta$ is not provable in $\mathbf{GS4}^{-}$.

- The set U_0 of all $\star$-consistent pairs is ordered by the relation $\leq$, where $(\Sigma, \Theta) \leq (\Lambda, \Pi)$ is defined by $\Sigma \subseteq \Lambda$ and $\Theta \subseteq \Pi$, as sets of formulas.

2. Constructing an **S4**-frame

Since U_0 is finite, trivially the following holds.

[Extension lemma] If (Σ_0, Θ_0) is $\star$-consistent then there is an extension (Σ, Θ) of (Σ_0, Θ_0) which is maximal $\star$-consistent.

Let U be the set of all maximal $\star$-consistent pairs. Of course, U is finite. Define $(\Sigma, \Theta) R (\Lambda, \Pi)$ by $\Sigma_\Box \subseteq \Lambda_\Box$ for $(\Sigma, \Theta), (\Lambda, \Pi) \in U$. Here $\Phi_\Box$ means the set $\{\gamma : \Box\gamma \in \Phi\}$. Obviously,

$\langle U, R \rangle$ is a (finite) **S4** frame, i.e., R is reflexive and transitive.

(Notice that $(\Sigma, \Theta) \in U$ doesn't necessarily imply that $\Sigma \cup \Theta = \Omega$.) Next, we show that *U is downward saturated*, i.e., every element $(\Sigma, \Theta) \in U$ satisfies the following conditions.

3. Downward saturation of U

0. For each propositional variable $p \in \Omega$, $p \notin \Sigma \cap \Theta$,

I. For each $\beta \wedge \gamma \in \Omega$,

- if $\beta \wedge \gamma \in \Sigma$ then both β and γ are in Σ,
- if $\beta \wedge \gamma \in \Theta$ then either β or γ is in Θ.

II. For each $\beta \vee \gamma \in \Omega$,

- if $\beta \vee \gamma \in \Sigma$ then either β or γ are in Σ,
- if $\beta \vee \gamma \in \Theta$ then both β and γ are in Θ.

III. For each $\neg\beta \in \Omega$,

- if $\neg\beta \in \Sigma$ then β is in Θ,
- if $\neg\beta \in \Theta$ then β is in Σ.

IV. For each $\Box\beta \in \Omega$,

- if $\Box\beta \in \Sigma$ then $\beta \in \Lambda$ for each $(\Lambda, \Pi) \in U$ such that $(\Sigma, \Theta)R(\Lambda, \Pi)$,
- if $\Box\beta \in \Theta$ then $\beta \in \Pi$ for some $(\Lambda, \Pi) \in U$ such that $(\Sigma, \Theta)R(\Lambda, \Pi)$.

4. Model existence lemma

If (Σ_0, Θ_0) is $\star$-consistent then there exists an assignment $\models$ on $\langle U, R \rangle$ such that every $\alpha \in \Sigma_0$ is true and every $\beta \in \Theta_0$ is false.

To show our model existence lemma, let us define a valuation $\models$ on the **S4** frame $\langle U, R \rangle$ by $(\Lambda, \Pi) \models p$ iff $p \in \Lambda$, for each $(\Lambda, \Pi) \in U$. With the help of downward saturation, we can show by induction that for every (Λ, Π)

- if $\gamma \in \Lambda$ then $(\Lambda, \Pi) \models \gamma$,
- if $\delta \in \Pi$ then $(\Lambda, \Pi) \not\models \delta$.

Applying this to a maximal $\star$-consistent extension (Σ, Θ) of (Σ_0, Θ_0), we have our lemma. (Such a valuation as above is called a *partial valuation* by K. Schütte.) In this way, we get a proof of cut elimination for **GS4**.

As for a general exposition of proof theory, please consult

- HO, Proof-theoretic methods in nonclassical logic – an introduction, Theories of Types and Proofs (MSJ Memoirs 2), Mathematical Society of Japan, 1998, pp.207-254.

- HO, *Proof Theory and Algebra in Logic*, Short Textbooks in Logic vol.2, Springer, 2019

♣ A historical note on semantical approaches to cut elimination

- S. Kanger, *Provability in Logic*, Stockholm Studies in Philosophy 1, 1957.
- H. Rasiowa and R. Sikorski, On the Gentzen theorems, (1959).
- K. Schütte, Syntactical and semantical properties of simple type theory, (1960).
- M. Fitting, Model existence theorems for modal and intuitionistic logics, (1973).
- M. Sato, A study of Kripke-type models for some modal logics by Gentzen's sequential method, (1977).

♣ Also on algebraic approaches to cut elimination

- A.G. Dragalin, *Mathematical Intuitionism: Introduction to Proof Theory*, AMS, 1988. — intuitionistic predicate logic.
- S. Maehara, Lattice-valued representation of the cut elimination theorem, Tsukuba J. of Math. 15 (1991). — semi-algebraic proof for classical and intuitionistic simple type theories.
- M. Okada and K. Terui, The finite model property for various fragments of intuitionistic linear logic, JSL 64 (1999). — for intuitionistic linear logic.
- F. Belardinelli, P. Jipsen and HO, Algebraic aspects of cut elimination, Studia Logica 77 (2004). — purely algebraic proof for substructural and modal logics.

♣ (continued)

- A. Ciabattoni, N. Galatos and K. Terui, Algebraic proof theory for substructural logics: cut-elimination and completions, APAL 163 (2012). — more general algebraic proofs, in which the connection with MacNeille completions is explicated.

- HO, A uniform algebraic approach to cut elimination via semi-completeness, Philosophical Logic: Current Trends in Asia (2017). — algebraic presentation of Maehara's approach (1991), which clarifies relations between the approach using Kripke-semantics and algebraic one.

- T. Arai, Cut-eliminability in second order logic calculus, Annals of the Japan Association for Philosophy of Science (2018). — algebraic proof for classical second order logic, which can unify approaches by Takahashi-Prawitz and the one by Maehara (1991).