Proof Mining:
Extraction of Information from Proofs

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Overview of Content

Lecture I: General Introduction to the Unwinding of Proofs ('Proof Mining')

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Lecture II: Functional Interpretation and Logical Metatheorems

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Lecture II: Functional Interpretation and Logical Metatheorems

Lecture III: Applications in Ergodic Theory, Convex Optimization and Nonlinear Semigroup Theory

- During (mainly) the last 20 years modern forms of functional interpretations were used to obtain **numerous new quantitative results** as well as **qualitative uniformity results** in particular in: nonlinear analysis, fixed point theory, ergodic theory, topological dynamics, approximation theory, convex optimization, abstract Cauchy problems, probability theory ($\geq$ 200 papers mostly in journals in the resp. areas or general mathematics journals).

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- General **logical metatheorems** explain this as instances of logical phenomena (K. APAL 1993, TAMS 2005, Gerhardy/K. TAMS 2008, Leucstean APAL 2014, Günzel/K. Adv. Math. 2016, Engracia/ Ferreira 2020, Sipoș MLQ 2023, K./Pischke Phil. Trans. Royal Soc. 2023, Pischke JML 2023, TAMS 2024, Neri/Pischke 2024.

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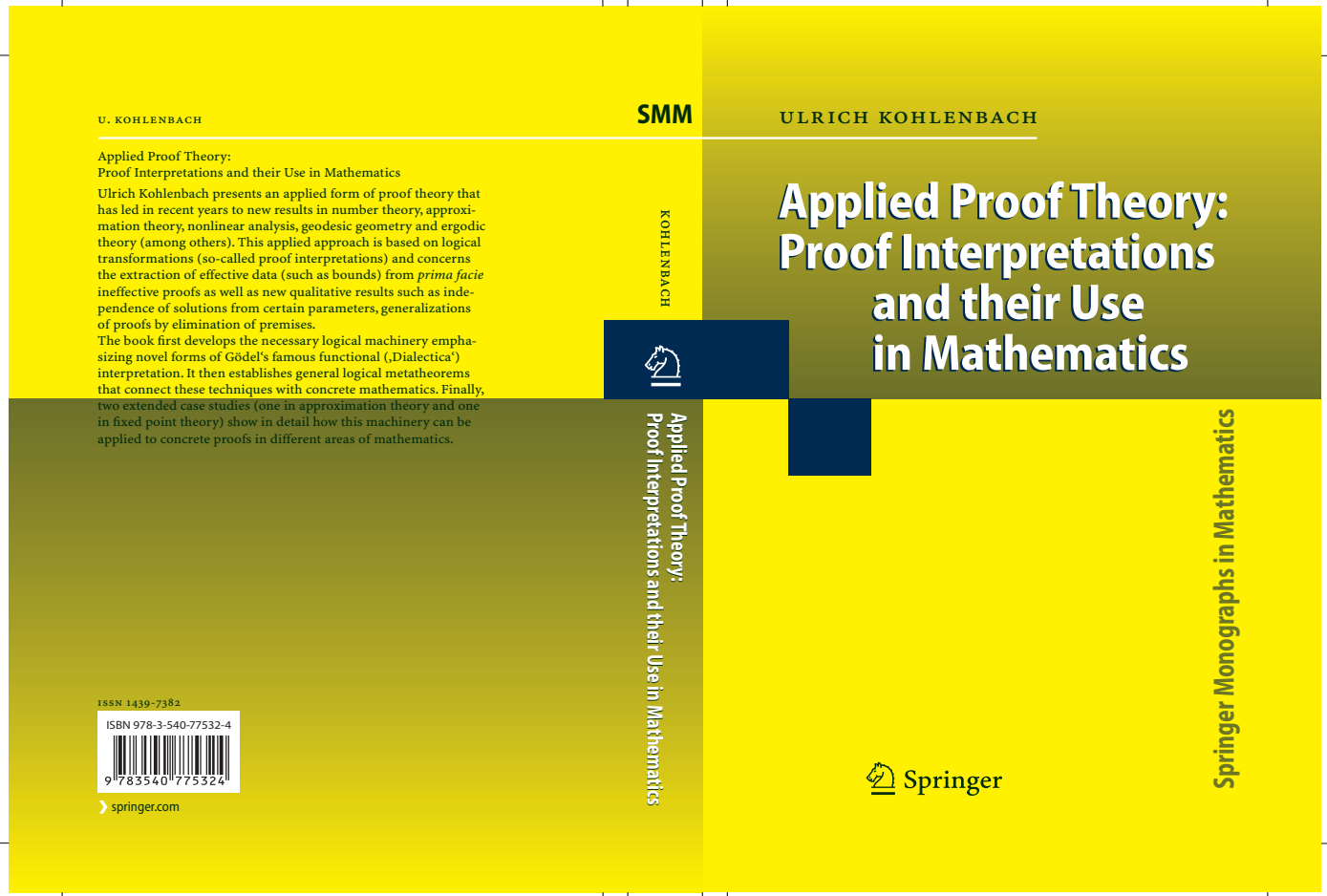
- Some of the logical tools used have been rediscovered in 2007 in special cases by Terence Tao prompted by concrete mathematical needs **'finitary analysis'!**

Main references

2008 book:



Recent surveys

1) U. Kohlenbach: Recent progress in proof mining in nonlinear analysis. IFCoLog J. of Logics and its Applications 10:4, pp. 3357-3406 (2017).

2) U. Kohlenbach: Proof-theoretic methods in nonlinear analysis. Proc. Int. Cong. of Math. - ICM 2018.

Both available at: www.mathematik.tu-darmstadt.de/~kohlenbach

Lecture I

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For '**real statements**' φ (i.e. $\varphi \in \Pi^0_1$) this would imply (Hilbert 1927)

$$\mathcal{T} \vdash \varphi \overset{\text{Kreisel}}{\Rightarrow} \mathrm{FIN} + \mathrm{Con}_{\mathcal{T}} \vdash \varphi \overset{\mathrm{FIN} \vdash \mathrm{Con}_{\mathcal{T}}}{\Rightarrow} \mathrm{FIN} \vdash \varphi.$$

- Though in this specific form in general impossible (Gödel), large parts of ordinary mathematics can be carried out in systems which can be proof-theoretically reduced to (and are conservative over) Peano arithmetic PA or even PRA (Feferman, Friedman, Simpson...).

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- While it has been claimed (Tait) that Hilbert's finitism corresponds to primitive recursive arithmetic PRA, Hilbert himself argued in 1925 for the finitistic character of the primitive recursive functionals of finite type (nowadays called Gödel's T) which includes the Ackermann function.
- In his 'Dialectica' paper (1958), Gödel showed that the consistency of Peano Arithmetic PA can be finitistically reduced to that of T.

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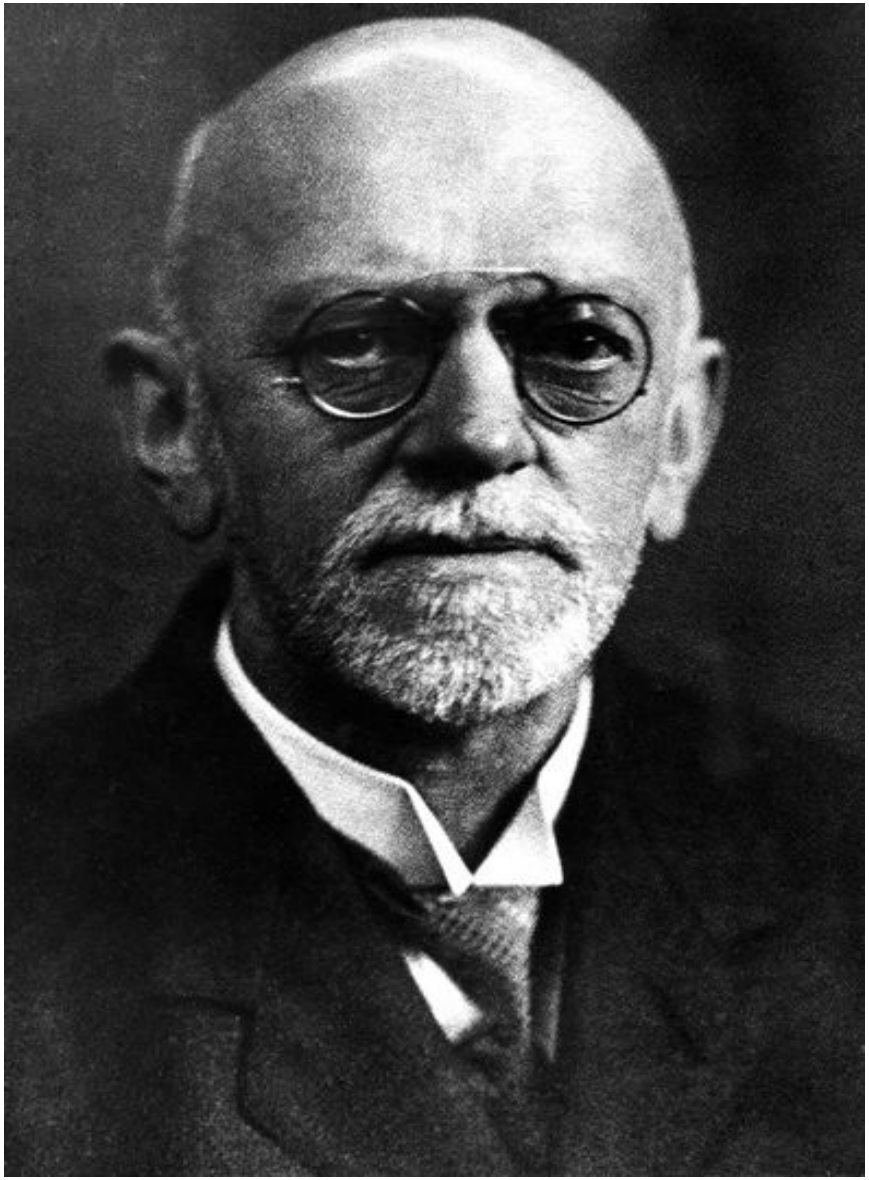
- Already Hilbert apparently had broader goal: exhibit the computational content of proofs of concrete statements (e.g. Π_2-sentences) which use ideal elements.
- In the Hilbert school bounds were extracted from proofs in abstract algebra, e.g. by Grete Hermann in her 1925 PhD thesis (supervised by Emmy Noether).

David Hilbert (1862-1943)



- Since the 50's, Kreisel suggested to use proof transformations developed in the course of Hilbert's program to analyze noneffective proofs in mathematics:

  Shift of emphasis (G. Kreisel $\geq$ 1951): use proof theory to **extract new information** from proofs of **existential** statements.

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- Early success: Luckhardt's polynomial bounds on Roth's theorem (1985, see below) and Delzell's work (1980-1996) on Hilbert's 17th problem.

Georg Kreisel (1923-2015)

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- quantitative information: explicit effective bounds,
- qualitative aspects:
  - existence of bound that is independent from certain parameters,
  - generalizations of proofs: weakening of premises,
  - finding 'correct' definitions of concepts.

Numerous case studies in nonlinear analysis show that indeed computational content of low complexity can be extracted even from proofs that use ideal principles:

proof-theoretic tameness of ordinary mathematics!

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- The finitistic use of the 'ideal elements' **replaces them by finitary versions** (without the proof noticing the difference!).
- This gives a **finitistic realization of $\forall\exists$-theorems** together with an elementary verification.
- For **semi-intuitionistic proofs** the restriction to $\forall\exists$-theorem can be overcome: **arbitrary logical complexity allowed!**

A polynomial rate of asymptotic regularity in Bauschke's solution of the 'minimal displacement conjecture'

Consider a Hilbert space H and nonempty closed and convex subsets $C_1, \ldots, C_N \subseteq H$ with metric projections P_{C_i}, define $T := P_{C_N} \circ \ldots \circ P_{C_1}$. In 2003 Bauschke proved the 'minimal displacement conjecture':

$$\| T^{n+1}x - T^n x\| \to 0 \quad (x \in H).$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^{N} C_i \neq \emptyset$) or C_i half spaces etc.

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Proof uses abstract theory of maximal monotone operators: Minty's theorem (Zorn's lemma), Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, Bruck-Reich theory of strongly nonexpansive mappings, conjugate functions, normal cone operator...).

Theorem (K. FoCM 2018)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, \quad D := 2b + NK, \quad \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon},$$

$K \geq \left(\sum_{i=1}^{N} \|c_i\|^2 \right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

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A. Sipoș: Extension to general **averaged mappings** (Opt.Lett. 2021) and to **super strongly nonexpansive mappings** (Israel J. Math. 2024).

Naive attempt fails

Naive Attempt: try to extract an explicit computable function realizing (or bounding) '$\exists y$': $\forall x \in \mathbb{N}\, F(x, f(x))$.

Proposition

There exist a sentence $A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

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Any bound g on '$\exists y$', i.e. no computable g such that

$$\forall x \exists y \leq g(x) \forall z\, (T(x, x, y) \vee \neg T(x, x, z))$$

since this would solve the halting problem! □

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Definition

$\boldsymbol{A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)}$. Then the **Herbrand normal form** of A is defined as

$$\mathbf{A^H :\equiv \exists x_1, x_2 A_{qf}(x_1, f(x_1), x_2, g(x_1, x_2))},$$

where $\boldsymbol{f, g}$ are new function symbols, called index functions.

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A and A^H are equivalent with respect to logical validity, i.e.

$$\models \mathbf{A} \Leftrightarrow \models \mathbf{A^H},$$

but are not logically equivalent (but only in the presence of AC).

We now consider again the sentence

$$\mathbf{A} \equiv \forall \mathbf{x}\, \exists \mathbf{y}\, \forall \mathbf{z}\, (\mathbf{P}(\mathbf{x}, \mathbf{y}) \vee \neg \mathbf{P}(\mathbf{x}, \mathbf{z})),$$

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$$\boldsymbol{A^{H,D}} :\equiv \big(\boldsymbol{P(x, c)} \vee \neg \boldsymbol{P(x, g(c))}\big) \vee \big(\boldsymbol{P(x, g(c))} \vee \neg \boldsymbol{P(x, g(g(c)))}\big)$$

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$$\boldsymbol{A^{H,D}} :\equiv \underbrace{\big(\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{c}) \vee \neg \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{c}))\big)}_{\in \mathrm{TAUT}} \vee \big(\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{c})) \vee \neg \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{g}(\boldsymbol{c})))\big)$$

is a tautology.

J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)$. Then:
PL $\vdash A$ iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

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is a tautology. $\boldsymbol{A^{H,D}}$ is called a **Herbrand Disjunction**.

Note that the length of this disjunction is fixed: $k \cdot n$. The terms s_i, t_j can be extracted from a given PL-proof of A.

Herbrand's Theorem continued

Replacing in $A^{H,D}$ all terms '$g(s_i, t_j)$', '$f(s_i)$', by new variables (treating larger terms first) results in another tautological disjunction A^{Dis} s.t. A can be inferred from A by a **direct proof**.

Remark

- For sentences $A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)$, A^{Dis} can be written in the form

$$A_{qf}(x, t_1, b_1) \vee A_{qf}(x, t_2, b_2) \vee \ldots \vee A_{qf}(x, t_k, b_k),$$

where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).

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  where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).
- Herbrand's theorem immediately extends to first-order theories $\mathcal{T}$ whose non-logical axioms $G_1, \ldots, G_n$ are all purely universal.

Theorem (Roth 1955)

An algebraic irrational number α has only finitely many exceptionally good rational approximations, i.e. for $\varepsilon > 0$ there are only finitely many $q \in \mathbb{N}$ such that

$$\mathbf{R(q) :\equiv q > 1 \wedge \exists! p \in \mathbb{Z} : (p, q) = 1 \wedge |\alpha - pq^{-1}| < q^{-2-\varepsilon}.}$$

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Theorem (Luckhardt 1985/89)

The following upper bound on $\#\{q : R(q)\}$ holds:

$$\#\{q : R(q)\} < \frac{7}{3}\varepsilon^{-1} \log N_\alpha + 6 \cdot 10^3 \varepsilon^{-5} \log^2 d \cdot \log(50\varepsilon^{-2} \log d),$$

where $N_\alpha < \max(21 \log 2h(\alpha), 2 \log(1 + |\alpha|))$ and h is the logarithmic absolute homogeneous height and $d = deg(\alpha)$.

Independently: Bombieri and van der Poorten 1988.

Exercise (U. Berger)

Consider open theory $\mathcal{T} := \{\forall x(S(x) \neq 0)\}$ in language with equality, constant 0 and two unary function symbols S, f.

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Proposition

$\mathcal{T} \vdash \exists x\big(f(S(f(x))) \neq x\big)$.

Proof: Suppose that

$$\forall x\big(f(S(f(x))) = x\big),$$

then f is injective, but also (since $S(x) \neq 0$) surjective on $\{x : x \neq 0\}$ and hence non-injective. Contradiction! □

Exercise (U. Berger)

Consider open theory $\mathcal{T} := \{\forall x(S(x) \neq 0)\}$ in language with equality, constant 0 and two unary function symbols S, f.

Proposition

$\mathcal{T} \vdash \exists x\big(f(S(f(x))) \neq x\big)$.

Proof: Suppose that

$$\forall x\big(f(S(f(x))) = x\big),$$

then f is injective, but also (since $S(x) \neq 0$) surjective on $\{x : x \neq 0\}$ and hence non-injective. Contradiction! □

Exercise 1: Analyze the above proof to extract Herbrand terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ s.t.

$$\mathrm{PL} \vdash \Big(\bigwedge_{i=1}^{k} S(s_i) \neq 0\Big) \to \bigvee_{j=1}^{n} \big(f(S(f(t_j))) \neq t_j\big).$$

Limitations

- Techniques work only for restricted formal contexts: mainly purely universal ('algebraic') axioms, restricted use of induction, no higher analytical principles.

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- Require that one can 'guess' the correct Herbrand terms: in general procedure results in proofs of length $2_n^{|P|}$, where $2_{n+1}^k = 2^{2_n^k}$ (n cut complexity).

Towards generalizations of Herbrand's theorem

Allow **functionals $\Phi(x, f)$** instead of just Herbrand terms: Let's consider again the example

$$\mathbf{A} \equiv \forall \mathbf{x} \exists \mathbf{y} \forall \mathbf{z} \big(\mathbf{T(x, x, y)} \vee \neg \mathbf{T(x, x, z)} \big) \big).$$

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A^H can be realized by a computable functional of type level 2 which is defined by cases:

$$\Phi(\mathbf{x}, \mathbf{g}) := \begin{cases} c & \text{if } \neg T(x, x, g(c)) \\ g(c) & \text{otherwise.} \end{cases}$$

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$$\Phi(x, g) := \begin{cases} c & \text{if } \neg T(x, x, g(c)) \\ g(c) & \text{otherwise.} \end{cases}$$

From this definition it easily follows that

$$\forall x, g \big(\mathbf{T}(x, x, \Phi(x, g)) \vee \neg \mathbf{T}(x, x, g(\Phi(x, g)))\big).$$

Φ satisfies **G. Kreisel's no-counterexample interpretation!**

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

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Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} \forall \mathbf{i,j} \in \mathbf{[n;n+m]}\ (|\mathbf{a_i} - \mathbf{a_j}| \leq \mathbf{2^{-k}}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.

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where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.
Then the (partial) Herbrand normal form of this statement is

$$\mathbf{(2)}\ \forall \mathbf{k} \in \mathbb{N} \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{i,j} \in \mathbf{[n;n+g(n)]}\ (|\mathbf{a_i} - \mathbf{a_j}| \leq \mathbf{2^{-k}}).$$

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By contrast, there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (also referred to as **'metastability'** by T.Tao):

Proposition

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Phi^*(g,k) \forall i,j \in [n; n+g(n)] \, (|a_i - a_j| \leq 2^{-k}),$$

where

$$\Phi^*(g,k) := \tilde{g}^{(2^k-1)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Remark

The previous result can be viewed as a polished form of a **Herbrand disjunction** of **variable (in k) length**:

$$\bigvee_{i=0}^{2^k-1} \left(|a_{\tilde{g}^{(i)}(0)} - a_{\tilde{g}(\tilde{g}^{(i)}(0))}| \leq 2^{-k}\right).$$

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Corollary (T. Tao's finite convergence principle)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 1 \geq a_0 \geq \ldots \geq a_M \geq 0 \exists N \in \mathbb{N}$$
$$\left(N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})\right).$$

One may take $M := \tilde{g}^{(2^k)}(0)$.

The Mean Ergodic Theorem

X **Hilbert space**, $f : X \to X$ **linear** and $\|f(x)\| \leq \|x\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad \mathbf{(n \geq 0)}$$

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)]\, \big(\|A_i(x) - A_j(x)\| < \varepsilon\big),$$

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$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^{(K)}(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
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Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner TAMS 2010 (again based on logical metatheorem).

Some history:

- 2004, first rate of metastability (for the asymptotic regularity of asymptotically nonexpansive mappings) extracted by Kohlenbach/Lambov).
- 2005, rate of metastability for Krasnoselski-Mann iterations of nonexpansive selfmappings $T : X \to X$ of compact hyperbolic spaces X (Kohlenbach).
- 2007, Tao introduced the term 'metastability' in connection with the von Neumann Mean Ergodic Theorem (MET).
- 2007, independently from Tao, the first rate of asymptotic regularity for MET was extracted by Avigad/Gerhardy/Towsner.
- 2008, Kohlenbach/Leustean generalized this with a better bound to uniformly convex Banach spaces which, subsequently, led to oscillation bounds by Avigad/Rute (see below).
- Since then, numerous papers extracting explicit rates of metastability have appeared.

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'We shall establish Theorem 1.6 by "finitary ergodic theory" techniques, reminiscent of those used in [Green-Tao]...' 'The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit'...'In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation'
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

No-Counterexample Interpretation (Kreisel 1951)

Recall: for a formula

$$A \equiv \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \, A_{qf}(x_1, y_1, \ldots, x_n, y_n)$$

we call a tuple of functionals $\underline{\varphi}$ a **solution for the no-counterexample interpretation of** A if $\underline{\varphi}$ provides a witness for A^H

$$\forall f_1, \ldots, f_n \, \exists x_1, \ldots, x_n \, A_{qf}(x_1, f_1(x_1), \ldots, x_n, f_n(x_1, \ldots, x_n)),$$

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i.e.

$$\forall \underline{f} \, A_{qf}(\varphi_1(\underline{f}), f_1(\varphi_1(\underline{f}), \ldots, \varphi_n(\underline{f}), f_n(\varphi_1(\underline{f}), \ldots, \varphi_n(\underline{f}))).$$

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

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Direct example: Infinitary Pigeonhole Principle (IPP):

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Related problem: bad behavior w.r.t. modus ponens!

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

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Lecture II

Detour through intuitionistic systems and higher types

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HA$^\omega$ has λ-abstraction $(\lambda x^\rho . t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$R_\rho(0, y, z) =_\rho y, \; R_\rho(x+1, y, z) =_\rho z(R_\rho xyz, x),$$

where $=_\rho$ is defined as pointwise (extensional) equality.

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PA$^\omega$ $=$**HA$^\omega$** $+ (A \vee \neg A)$.

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where $=_\rho$ is defined as pointwise (extensional) equality.

PA$^\omega$ $=$**HA$^\omega$** $+ (A \vee \neg A)$.

Only **quantifier-free rule of extensionality** (A_{qf} quantifier-free)

$$\frac{A_{qf} \to s =_\rho t}{A_{qf} \to r[s] =_\tau r[t]}.$$

$\mathcal{S}^\omega$ full set-theoretic model of **PA$^\omega$**.

In 2007 is het 100 jaar geleden dat L.E.J. Brouwer (1881 – 1966) de stelling van Aristoteles verwierp. Brouwer vond dat een wiskundige stelling pas waar is als er 'Positief Bewijs' is. Brouwer is de grondlegger van de intuïtionistische wiskunde. Naar hem is o.a. de dekpuntstelling van Brouwer vernoemd. Iedere drie jaar reikt het Koninklijk Wiskundig Genootschap de Brouwer medaille uit aan een belangrijk wiskundige. Voor meer informatie: www.knaw.nl
NEDERLAND
€0,44
$\nvdash A \vee \neg A$
Er is positief bewijs!
100 jaar na dato wordt de wiskundige
L.E.J. Brouwer (1881-1966)
geëerd met een eigen postzegel.
TNT post

Proof mining in the intuitionistic case: modified realizability (Kreisel 1959)

To each $A \in \mathcal{L}(\mathrm{HA}^\omega)$ we assign a new formula $\underline{x}\ mr\ A$ ('$\underline{x}$ modified realizes A') inductively by

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(i) $\underline{x}\ mr\ A :\equiv A$ with the empty tuple $\underline{x}$, if A is a prime formula.

(ii) $\underline{x}, \underline{y}\ mr\ (A \wedge B) :\equiv \underline{x}\ mr\ A \wedge \underline{y}\ mr\ B$.

(iii) $z^{\mathbb{N}}, \underline{x}, \underline{y}\ mr\ (A \vee B) :\equiv [(z = 0 \to \underline{x}\ mr\ A) \wedge (z \neq 0 \to \underline{y}\ mr\ B)]$.

(iv) $\underline{y}\ mr\ (A \to B) :\equiv \forall \underline{x}(\underline{x}\ mr\ A \to \underline{y}\,\underline{x}\ mr\ B)$.

(v) $\underline{x}\ mr\ (\forall y^\rho A(y)) :\equiv \forall y^\rho(\underline{x}y\ mr\ A(y))$.

(vi) $z^\rho, \underline{x}\ mr\ (\exists y^\rho A(y)) :\equiv \underline{x}\ mr\ A(z)$.

Program extraction by modified realizability

Axiom of choice (in all types)

$$\mathrm{AC}: \quad \forall a^{\alpha}\, \exists b^{\beta}\, F(a,b) \to \exists B^{\rho\to\tau}\, \forall a^{\rho}\, F(a, B(a)).$$

Theorem

From a proof of

$$\mathrm{HA}^{\omega}+\mathrm{AC} \vdash \forall x^{\rho}\, (\neg B(x) \to \exists y^{\tau}\, A(x,y))$$

one can extract by *mr* a primitive recursive functional Φ s.t.

$$\mathcal{S}^{\omega} \models \forall x^{\rho}\, (\neg B(x) \to A(x, \Phi(x)))$$

(A, B, ρ, τ arbitrary).

Towards proofs based on classical logic

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Entrance door for classical logic: Markov's principle $\mathbf{M}^\omega$!

$$\mathbf{M}^\omega: \neg\neg\exists x^\rho A_{qf}(x) \to \exists x^\rho A_{qf}(x), \quad A_{qf} \text{ quantifier-free.}$$

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For $\rho \neq \mathbb{N}$: not even unbounded search possible!

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^{\omega}$ but eliminate it from proofs!

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Interpretation differs from mr for the clause of '$\to$'

$$(A \to B)^D :\equiv \exists \underline{U}\, \underline{Y}\, \forall \underline{x}\, \underline{v}\, \big(\underbrace{A_D(\underline{x}, \underline{Y}\,\underline{x}\,\underline{v}) \to B_D(\underline{U}\underline{x}, \underline{v})}_{(A \to B)_D :\equiv} \big).$$

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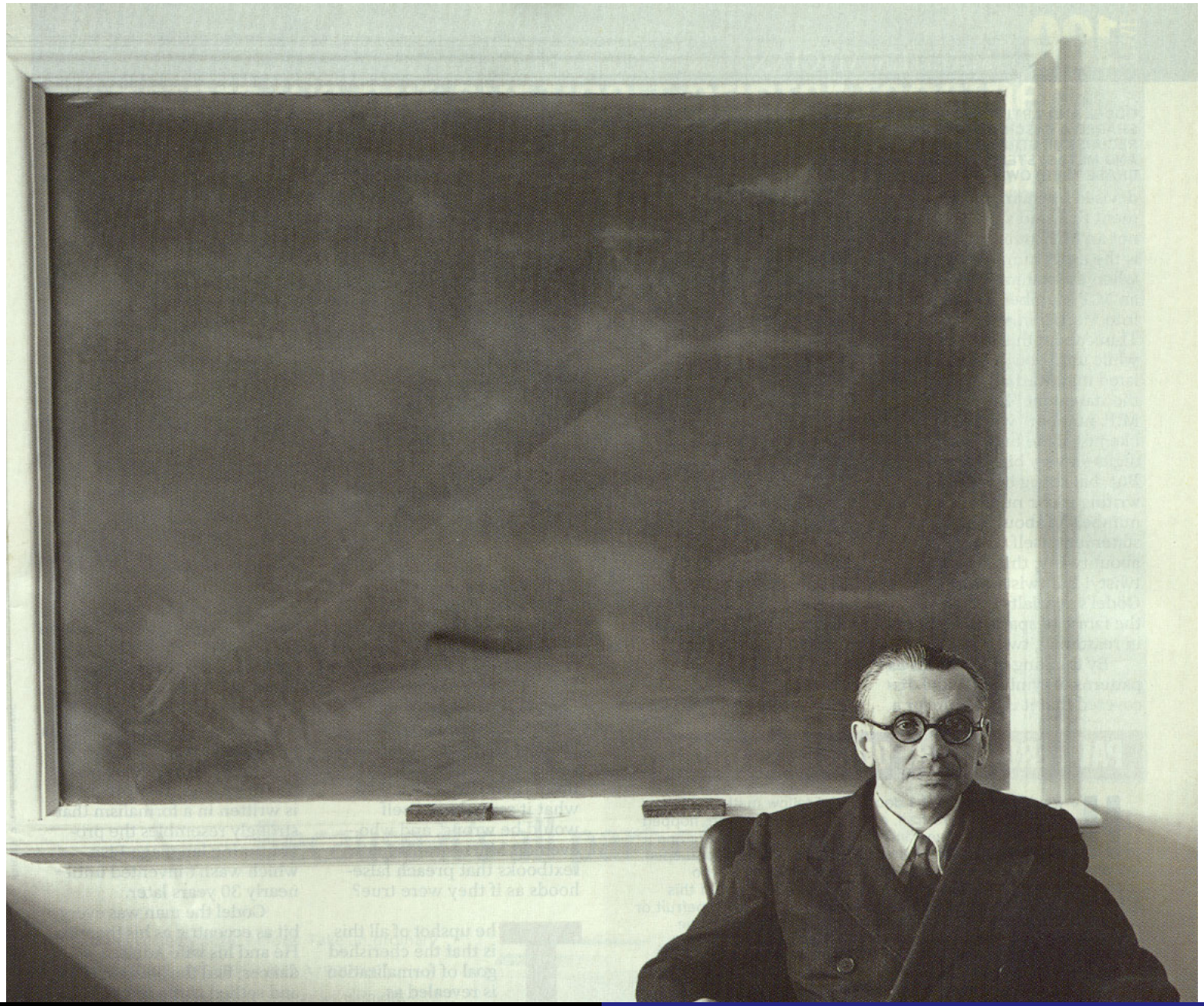
Partial alternative: Friedman-Dragalin A-interpretation after negative translation as intermediate step (variants: Schwichtenberg, Coquand-Hofmann).



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$$\mathbf{AC}: \ \forall a^{\alpha}\, \exists b^{\beta}\, A(a,b) \to \exists F^{\alpha\to\beta}\, \forall b^{\beta}\, A(a,Fa).$$

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Proofs based on full classical logic

Negative translation N combined with D (i.e. $D \circ N$) gives:

Theorem

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one can extract by *Sh* a primitive recursive functionals Φ s.t.

$$\mathcal{S}^{\omega} \models \forall x^{\rho} \left(\forall u\, B_{qf}(x,u) \rightarrow\ A_{qf}(x,\Phi(x))\right)$$

(ρ, τ arbitrary, A_{qf}, B_{qf} quantifier-free, QF-AC restriction of AC to quantifier-free formulas).

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$A \mapsto A^{Sh} \quad \text{(Shoenfield variant)}$$

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- $\exists Y \forall x\, A_{Sh}(x, Y(x)) \equiv A^{D \circ N}$.

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$\forall n, u, v \exists x, y\, (n{=}0 \to A_{Sh}(u,x)) \wedge (n{\neq}0 \to B_{Sh}(v,y))$
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Comments

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- It also scales up all the way to full countable and even dependent choice (including full 2nd order arithmetic), where then Φ is bar recursive: Spector 1962 (Consistency proof for analysis!).
- Full extensionality is allowed (since it can be eliminated) if all variables have types ≤ 1.

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^{ND}, A^{n.c.i}$ its Skolem, $D \circ N$ and $n.c.i.$ interpretations resp., then

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- $A^{n.c.i.}$ **too weak** (see IPP above; modus ponens problem).
- A^{ND} **just right**:

$$\mathbf{PA}^\omega\mathbf{+QF\text{-}AC} \vdash A \leftrightarrow A^{ND}.$$

Majorizability

The functionals occurring in functional interpretation have a striking mathematical structure property:

Definition (W.A. Howard 1973)

$$\begin{cases} x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x, \\ x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y(y^* \gtrsim_\rho y \to x^*(y^*) \gtrsim_\tau x(y)). \end{cases}$$

Read: 'x^* **majorizes** x' for $x^* \gtrsim x$.

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Monotone functional interpretation MD (K.96) directly extracts t^*.

Uniform bound extraction by NMD

Let Δ be a set of $\mathcal{S}^\omega$-valid sentences of the form

$$\forall a^\gamma \exists b \leq_\delta ta \forall c^\eta F_{qf}(a, b, c)$$

(γ, δ, η arbitrary, t closed term, F_{qf} quantifier-free).

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Example of axiom Δ: WKL!

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Example of axiom Δ: WKL!

Theorem (K., JSL 1992)

From a proof of

$$\mathbf{PA^\omega + QF\text{-}AC + \Delta} \vdash \forall x^{\mathbb{N}\to\mathbb{N}} \forall y \leq_\rho s x \exists z^{\mathbb{N}} \, A_{qf}(x,y,z)$$

one can extract by $MD \circ N$ a primitive recursive functionals Φ s.t.

$$\mathcal{S}^\omega \models \forall x^{\mathbb{N}\to\mathbb{N}} \forall y \leq_\rho s x \exists z \leq \Phi(x) \, A_{qf}(x,y,z).$$

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- Extraction of **bounds** from **noneffective** existence proofs.

In the uniform bound extraction theorem, the bound $\Phi(x)$ only depends on $x \in \mathbb{N}^{\mathbb{N}}$ but not on $y \leq_\rho sx$. For $\rho = \mathbb{N} \to \mathbb{N}$ and $s(x) :\equiv 1$ this, in particular, gives **independence from parameters in Cantor space** $2^{\mathbb{N}}$.

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Polish spaces X are represented as the quotient of the space of fast (e.g. 2^{-n}) convergent Cauchy sequences f of elements of a countable dense subset (f can be viewed as elements in $\mathbb{N}^{\mathbb{N}}$) w.r.t. equivalence relation

$$f =_X g :\equiv \lim_{n\to\infty} d_X(f(n), g(n)) =_{\mathbb{R}} 0.$$

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= **basic arithmetic**, HBC Heine/Borel compactness WKL (SEQ$^-$ restricted sequential compactness, ACA).

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Important:

$\Phi(f_x)$ does **not depend** on $y \in K$ but on a **representation** f_x of x!

All conditions necessary: total boundedness and completeness of K, A purely existential, Φ depending on name f_x of x.

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

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$$\forall x \in P, y_1, y_2 \in K \Big(\bigwedge_{i=1}^{2} f(x, y_i) =_{\mathbb{R}} 0 \to d_K(y_1, y_2) =_{\mathbb{R}} 0 \Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall x \in P, y_1, y_2 \in K, \varepsilon > 0 \Big(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon \Big).$$

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Theorem (K./Paulo Oliva, APAL 2003)

Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\Psi(\omega, n, \varepsilon) := \min\{\tfrac{c_n \varepsilon}{8(n+1)^2}, \tfrac{c_n \varepsilon}{2} \omega_n(\tfrac{c_n \varepsilon}{2})\}, \text{ where}$$
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Best Chebycheff approximation: K. 1990, Num.Funct.Optim. 1993.

Best Chebycheff approximation by polynomials with restricted coefficients: K. 1990, Sipoș Math. Nachrichten 2021.

The nonseparable/noncompact case: a simply example

Proposition

Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| = \text{dist}(x, C)$.

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Hence: if X is separable and complete and provably strictly convex and C compact, then one can extract a modulus of uniqueness.

Observation: compactness only used to extract uniform bound on strict convexity (= **modulus of uniform convexity**) from proof of strict convexity.

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Conclusion: neither compactness nor separability required!

In particular: **existence** of solution (for complete X and closed C) from **uniform uniqueness** which in turn stems from **uniform convexity**.

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, $\mathbb{R}$-trees, normed, uniformly convex, uniformly smooth, Hilbert, abstract L^p and $C(K)$-spaces, dual spaces, tracial von Neumann algebras, probability spaces, spaces of Rademacher type p : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

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Counterexamples (to extractibility of uniform bounds): for the classes of strictly convex ($\to$ uniformly convex) or separable ($\to$ totally bounded) spaces!

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathbf{PA}^{\omega,X}$ is the extension of Peano Arithmetic to all types.

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{A_{qf} \to s =_\rho t}{A_{qf} \to r(s) =_\tau r(t)},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$x^X =_X y^X :\equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}},$$
$$x =_{\rho\to\tau} y :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$x^{\mathbb{N}} \geq^{a}_{\mathbb{N}} y^{\mathbb{N}} :\equiv x \geq y$$
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Normed linear case: $a := 0_X$.

Theorem (Gerhardy/K.,Trans.Amer.Math.Soc. 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ types in $\{X, \mathbb{N} \to X, X \to X\}$ or a bit more complicated.
If $\mathcal{A}^\omega[X, \|\cdot\|]$ **proves**

$$\forall x \in P \forall y \in K \forall \underline{z}^{\underline{\tau}} \exists v^{\mathbb{N}} A_\exists(x, y, \underline{z}, v),$$

then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nontrivial normed space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\underline{z}^* \gtrsim_{\underline{\tau}} \underline{z}$):

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For the bounded cases: K. Trans.AMS 2005.

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

Corollary (Gerhardy/K., TAMS 2008)

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holds in **all nontrivial normed spaces**.

Admissible classes of functions: uniformly continuous, Lipschitzian, nonexpansive, firmly-, averaged- and strongly nonexpansive mappings; some classes of **discontinuous** functions: pseudo-contractions, maps with Suzuki's condition (E) etc.

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Recently: set-valued accretive operators and their resolvents in Cauchy problems and Optimization: (K./Koutsoukou-Argyraki, K./Powell, K./Pischke, Pischke, Pischke/Powell), gradients and Fenchel conjugates (Pischke), Daniel integrable functions (Neri/Pischke), super strongly nonexpansive mappings (Sipoș).

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Lecture III

As special case of **general logical metatheorems** one has:

Corollary (Gerhardy/K., TAMS 2008)

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holds in **all nontrivial normed spaces**.

Similar for Hilbert spaces and uniformly convex spaces (then bound depends on modulus of convexity).

Applications to Ergodic Theory

The Mean Ergodic Theorem

X **Hilbert space**, $f : X \to X$ **linear** and $\|f(z)\| \leq \|z\|$ for all $z \in X$.

$$\mathbf{A_n(z)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(z)}, \text{ where } \mathbf{S_n(z)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(z)} \quad (\mathbf{n \geq 0})$$

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Avigad/Gerhardy/Towsner (TAMS 2010):

in general **no computable rate of convergence**.

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

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$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^{(K)}(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
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Extension to finitely many mappings Sipoş (Pacific J. Math. 2021).

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

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As a corollary to our analysis of Birkhoff's proof, Avigad and Rute showed

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$(A_n(z))$ admits at most

$$2\log(M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma} \cdot (2\log(2M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma}$$

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For the Hilbert space case: first fluctuation bounds by Jones, Ostrovskii, Rosenblatt 1996.

Extensions to Nonlinear Ergodic Theorems

- **Metastability for Baillon's nonlinear ergodic theorem** (**K., Comm.Contemp.Math. 2012**).
- **Metastability for Wittmann's strong nonlinear ergodic theorem on Halpern iterations** (**K., Adv.Math. 2011**).
- **Generalization of the above to CAT(0)-spaces** (Saejung 2010) (**K., Leuştean: Adv.Math. 2012**).
- **Generalization to CAT(κ) spaces** $\kappa > 0$ (**Leuştean, Nicolae Ergodic Theor. Dynam. Syst. 2016**).
- **Metastable version of strong nonlinear ergodic theorem** for maps

$$\forall u, v \in C\ (\|f(u) + f(v)\| \leq \|u + v\|).$$

(covers Baillon's result for n.e. odd operators): **Safarik, J.Math.Anal.Appl. 2012**.

Let X be a uniformly convex Banach space, $C \subseteq X$ nonempty, bounded, closed and convex, $T : C \to C$ nonexpansive.

$$S_n x := \frac{1}{n} \sum_{i=0}^{n-1} T^i x.$$

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Bruck's proof crucially uses a result due to Pisier 1973 that uniformly convex Banach spaces (in fact uniformly nonsquare and even B-convex Banach spaces) have a nontrivial Rademacher type $q > 1$ which implies that there exists a constant C_q s.t. for every finite sequence $X_1, \ldots, X_n$ of independent mean zero Radon random variables in $L_q(X)$

$$\mathbb{E}\left(\left\|\sum_{i=1}^{n} X_i\right\|^q\right)^{1/q} \leq 2C_q \cdot \left(\sum_{i=1}^{n} \mathbb{E}\left(\|X_i\|^q\right)\right)^{1/q}.$$

Let $(X, \|\cdot\|)$ be **uniformly nonsquare** with $\delta \in (0, 1]$ such that

$$\min\left\{\frac{\|x-y\|}{2}, \frac{\|x+y\|}{2}\right\} \leq (1-\delta)\cdot \max\{\|x\|, \|y\|\}$$

holds for all $x, y \in X$.

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If X is uniformly convex and η is a modulus of convexity, we may take $\delta := \eta(1)$.

From Pisier's proof one can extract the following explicit constants:

Theorem (Pisier 1973, Freund/K. ASVAO 2022)

Let $(X, \|\cdot\|)$ be a uniformly nonsquare Banach space with $\delta \in (0,1)$ witnessing this property. Define $\lambda := 1 - \delta$. Assume that $\xi \in (0,1)$ is so small and that $p' \in [2, \infty)$ is so large that

$$\frac{1-\xi}{1+2\sqrt{2\xi}} \geq \frac{1}{2}\sqrt{2\lambda^2+2} \quad \text{and} \quad \frac{1}{2^{1/p'}} \geq 1-\xi.$$

Take p with $1 = \frac{1}{p} + \frac{1}{p'}$.

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Then - for any $q \in (1,p)$ - X has Rademacher type q with constant

$$C_q = 3 \cdot \frac{2^{1/q}}{2^{(1/q)-(1/p)} - 1}.$$

Analyzing proofs due to Bruck 1979, 1981 one can extract:

Theorem (Freund/K. ASVAO 2022)

Let $(X, \|\cdot\|)$ be uniformly convex with modulus η. $T : C \to C$ nonexpansive, $C \subseteq B_{b/2}(0)$ closed and convex. Assume X is of Rademacher type $q \in (1, 2]$ with constant C_q. Given $\varepsilon \in (0, 1]$, pick $\tilde{p} \in \mathbb{N}$ so large that $2C_q \cdot \tilde{p}^{(1-q)/q} \leq \varepsilon/(9b)$. Consider $p \in \mathbb{N}$ with $p \geq 2b/\delta^2$ for

$$\delta := \xi^{\tilde{p}}\left(\frac{\varepsilon}{9}\right) \quad \text{with} \quad \xi(t) := \frac{t}{12} \cdot \eta\left(\min\left\{2, \frac{t}{b}\right\}\right).$$

For any $\alpha < \xi^{p-1}(\delta^2/2)$ with $0 < \alpha < \varepsilon/3$ we have

$$\forall n \geq \frac{b}{\alpha} \ \forall x \in C \ (\|T(S_n x) - S_n x\| \leq \varepsilon).$$

Theorem (Kobayasi-Miyadera 1980)

Consider $T : C \to C$ as above. Given $x \in C$, assume that the sequences $(\|T^n x - T^{n+i} x\|)_n$ converge uniformly in i. We then have

$$\lim_{n\to\infty} \|y - S_n T^k x\| = 0 \quad \text{uniformly in } k,$$

for some fixed point y of T.

Freund/K. (ETDS 2023): Explicit function Φ which translates any rate A of metastability for the assumption that the sequences $(\|T^n x - T^{n+i} x\|)_n$ converge uniformly in i into a common rate of metastability for the conclusion.

Proof mining involving monotone operators

Maximal monotone operators

Let $(\lambda_n) \subset (0, \infty)$ and $A : X \to 2^X$ a set-valued (maximal) monotone (resp. accretive) operator with$X$ Hilbert (resp. Banach space):

$$\langle x - y, u - v \rangle \geq 0, \ u \in Ax, v \in Ay$$

(in Banach space: normalized duality map instead of scalar product).

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- Important theorems on such operators such as the **Brezis-Haraux theorem** admit a **tame** proof-theoretic analysis (K./Pischke, Phil. Trans. Royal. Soc. 2023)

- The fundamental relation $(+)\ 0 \in Az \leftrightarrow z =_X J_A z$ (which needs extensionality) holds quantitatively if $0 \in Az$ is replaced by

$$\exists x, \tilde{z} \in X\,(x =_X 0 \wedge z =_X \tilde{z} \wedge x \in A\tilde{z})$$

via the provable intensional certificate for '$\leftarrow$' in $(+)$

$$z - J_A z \in A(J_A z)$$

and the provable

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Always **strong convergence** for **Halpern type variant of PPA**:

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) J_{\lambda_n A} x_n \quad \textbf{(HPPA)}$$

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That analysis is based on the mining (due to K./Sipoș 2021) of a proof of a deep theorem by Reich for uniformly smooth spaces.

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- CAT(0)-spaces (Sipoș Comp.Optim.Appl. 2022)
- maximally monotone operators $A : X \to 2^{X^*}$ in Banach spaces with **Bregman resolvents** (Pischke/K.2023)
- a **new class** of **uniformly smooth hyperbolic spaces** (Pinto TAMS 2024). That extension of 'uniformly smooth' to the metric setting was motivated by the aforementioned mining of Reich's theorem which has been extended by Pinto to the metric setting.

A rate of asymptotic regularity for the Bregman HPPA

X real Banach space, $f : X \to \mathbb{R}$ supercoercive, bounded and totally convex on bounded sets and uniformly Fréchet differentiable and f^* is the Fenchel conjugate. $A : X \to 2^{X^*}$ maximally monotone with $A^{-1}0 \neq \emptyset$. $(\alpha_n) \subset (0, 1]$ with $\lim \alpha_n = 0, \sum \alpha_n = \infty$, $(r_n) \subset (0, \infty)$ with $\inf r_n > 0$. For $x_0 = x, u \in X$

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Pischke 2023: extensions of logical metatheorem which include all the above concepts and cover these applications!

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Definition

A sequence (x_n) in a metric space (X, d) is Fejér monotone w.r.t. a subset $F \subseteq X$ if $\forall n \in \mathbb{N} \forall p \in F \ (d(x_{n+1}, p) \leq d(x_n, p))$.

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Why is this important?

If one has metric regularity one not only gets strong convergence but even a **rate of convergence!**

Moduli of regularity for mappings

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Definition (K./Lopéz-Acedo/Nicolae, Israel J. Math 2019)

Let $zer\, F := \{x \in X : F(x) = 0\} \neq \emptyset$. F is **regular** w.r.t. $zer\, F$ if

$$\forall n \in \mathbb{N} \exists k \in \mathbb{N}\, \forall x \in X \big(|F(x)| < 2^{-k} \rightarrow \exists z' \in zer\, F (d(x, z') < 2^{-n})\big)\,.$$

A function $\omega : \mathbb{N} \rightarrow \mathbb{N}$ providing $k = \omega(n)$ is a **modulus of regularity**.

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A function $\omega : \mathbb{N} \to \mathbb{N}$ providing $k = \omega(n)$ is a **modulus of regularity**.

Covers: fixed point, equilibrium problems, zeros of set-valued maps.

Computational use of moduli of regularity

Proposition (K./Lopéz-Acedo/Nicolae Israel J. Math. 2019)

Let $F : X \to \mathbb{R}$ *be with* $\operatorname{zer} F \neq \emptyset$ *and with modulus of metric regularity* ω. *Let* (x_n) *be a sequence in* X *and* $\psi : \mathbb{N} \to \mathbb{N}$ *be s.t.*

$$\forall k \in \mathbb{N} \exists n \leq \psi(k) \ \left(|F(x_n)| < 2^{-k}\right),$$

where (x_n) *is Fejér monotone w.r.t.* $\operatorname{zer} F$. *Then* (x_n) *is Cauchy:*

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Computational use of moduli of regularity

Proposition (K./Lopéz-Acedo/Nicolae Israel J. Math. 2019)

Let $F : X \to \mathbb{R}$ *be with* $\operatorname{zer} F \neq \emptyset$ *and with modulus of metric regularity* ω. *Let* (x_n) *be a sequence in* X *and* $\psi : \mathbb{N} \to \mathbb{N}$ *be s.t.*

$$\forall k \in \mathbb{N} \exists n \leq \psi(k) \ \left(|F(x_n)| < 2^{-k}\right),$$

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If X *is complete and* F *is continuous, then* $\lim x_n \in \operatorname{zer} F$.

Dykstra's algorithm and generalized Fejér monotonicity

Dykstra algorithm: strongly convergent variant of the weakly convergent cyclic projection method to compute points in intersections $C := \bigcap_{i=1}^{m} C_i$ of closed convex subsets in Hilbert space (consistent feasibility problem).

$$\begin{cases} x_0 \in X \\ q_{-(m-1)} = \ldots = q_0 = 0 \end{cases} \text{ and } \begin{cases} x_n := P_{C_{j(n)}}(x_{n-1} + q_{n-m}) \\ q_n := x_{n-1} + q_{n-m} - x_n, \end{cases}$$

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- effective rate of metastability,
- Although (x_n) is **not Fejér monotone** (w.r.t. C): effective rate of convergence in the presence of a modulus of regularity.
- This lead to a **new generalized form** of **Fejér monotonicity** which covers this (K./Pinto 2024).

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In general, **no computable** moduli of metric regularity:

Proposition (K./López-Acedo/Nicolae Israel J. Math. 2019)

There exists a **computable firmly nonexpansive** mapping $T : [0,1] \to [0,1]$ which has **no computable modulus** of metric regularity ϕ w.r.t. $Fix(T)\ (= zer\ (I - T))$.

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In fact, the cases where one can compute such a modulus are rare. However there are important cases where this is true (connection to o-minimality: tame optimization, Ioffe, Lewis, Bolte, Daniilidis...!)

Linear rates for the Tikhonov regularization method

Recently, Boţ et al. studied **strongly convergent** iteration methods in optimization using a so-called **Tikhonov regularization term** which has been generalized by Cheval/Leuştean (Optimiz. Method. Softw., to appear) to the **Tikhonov-Mann iteration**

$$\text{(T-M)}: \begin{cases} x_{n+1} = (1-\lambda_n)u_n + \lambda_n T u_n \\ u_n = (1-\beta_n)u + \beta_n x_n, \end{cases}$$

where $C \subseteq X$ is a convex subset of a W-hyperbolic space X, $T : C \to C$ is **nonexpansive** and $(\lambda_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0,1]$ and $x_0, u \in C$ and is studied in proof mining also by Dinis/Pinto.

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While Boţ et al. only considered the case $u := 0$ in Hilbert spaces, (x_n) is defined in arbitrary W-hyperbolic spaces.

A lemma due to Sabach/Shtern 2017 combined with the results obtained by proof mining in the general case gives a **linear** rate of asymptotic regularity for $\lambda_n = \lambda \in (0,1), \beta_n := 1 - \frac{2}{n}$ in general hyperbolic spaces:

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Theorem (Cheval/K./Leuștean, JOTA 2023)

Let $K \geq \max\{d(x_0,p), d(u,p)\}$ for some fixed point p of T. Assume that $\lambda_n = \lambda \in (0,1]$, $\beta_1 = 0$ and $\beta_n = 1 - \frac{2}{n}$ for $n \geq 2$, and let (x_n) be the Tikhonov-Mann iteration. Then for all $n \geq 1$,

$$d(x_{n+1}, x_n) \leq \frac{4K}{n+1}, \; d(x_n, Tx_n) \leq \frac{8K}{\lambda n}.$$

(T-M) has been generalized by Dinis/Pinto to two nonexpansive mappings T, U

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Theorem (Dinis/Pinto, SIAM Journal on Optimization 2023)

If U, T have common fixed points, then (x_n) strongly converges to the common fixed point closest to u if X is a CAT(0) space under suitable conditions on the scalars. Moreover, an explicit effective rate of metastability can be given.

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For $U := Id$ the result is due to Seajung 2010 and - quantitatively - Schade/K. 2012.

A polynomial rate of asymptotic regularity in Bauschke's solution of the 'minimal displacement conjecture'

Consider a Hilbert space H and nonempty closed and convex subsets $C_1, \ldots, C_N \subseteq H$ with metric projections P_{C_i}, define $T := P_{C_N} \circ \ldots \circ P_{C_1}$. In 2003 Bauschke proved the 'minimal displacement conjecture':

$$\| T^{n+1}x - T^n x\| \to 0 \quad (x \in H).$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^{N} C_i \neq \emptyset$) or C_i half spaces etc.

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Proof uses abstract theory of maximal monotone operators: Minty's theorem (Zorn's lemma), Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, Bruck-Reich theory of strongly nonexpansive mappings, conjugate functions, normal cone operator...).

$(\|T^{n+1}x - T^n x\|)_{n\in\mathbb{N}}$ is **nonincreasing** and hence the conclusion in Bauschke's theorem is of the form $\forall\exists$.

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Logical metatheorems, therefore, guarantee the extractability a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$ and **majorants** for $x \in H$ and $P_{C_1}, \ldots, P_{C_N}$:

$b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$:

$$\|P_{C_i}0\| \leq \|c_i\| \leq K.$$

Since the mappings P_{C_i} are nonexpansive, the corollary guarantees a computable $\Phi(\varepsilon, N, b, K)$ s.t. for $b \geq \|x\|$

$$\forall\varepsilon > 0\, \forall n \geq \Phi(\varepsilon, N, b, K)\ (\|T^{n+1}x - T^n x\| < \varepsilon).$$

This explains the rate presented in Lecture I!

Proof Mining in Nonlinear Semigroup Theory

Let X be a Banach space, $C \subseteq X$ convex, $\lambda \in (0,1)$.

Definition: A family $\{T(t) : t \geq 0\}$ of nonexpansive mappings $T(t) : C \to C$ is a **nonexpansive semigroup** if

(*i*) $\boldsymbol{T(s+t) = T(s) \circ T(t) \quad (s,t \geq 0)}$,

(*ii*) for each $\boldsymbol{x \in C}$, the mapping $\boldsymbol{t \mapsto T(t)x}$ is continuous.

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Theorem (Suzuki, LMS 2006) Let $0 < \alpha < \beta$ such that α/β is irrational. Then any fixed point of $p \in C$

$$S := \lambda T(\alpha) + (1-\lambda) T(\beta)$$

is a common fixed point of $T(t)$ for all $t \geq 0$, i.e.

$$Fix(S) \subseteq \bigcap_{t \geq 0} Fix(T(t)).$$

General logical metatheorems guarantee:

let $t \mapsto T(t)x$ be equicontinuous on norm-bounded subsets of X with modulus ω, let f_γ be a **modulus of irrationality** for $\gamma := \alpha/\beta$, $\Lambda, N, D \in \mathbb{N}$ be s.t. $1/\Lambda \leq \lambda, 1-\lambda$ and $1/N \leq \beta \leq D$.

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$$\|p\| \leq d \wedge \|S(p) - p\| \leq \Phi(\varepsilon, M, d) \rightarrow \forall t \in [0, M] \, (\|T(t)p - p\| \leq \varepsilon).$$

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Let $x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Sx_n$ be a d-bounded Krasnoselski iteration of S with rate of asymptotic regularity Ψ, then

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Note: Suzuki's proof uses WKL: **WKL-elimination!**

Theorem (Koutsoukou-Argyraki/K., J.Math.Anal.Appl. 2016)

Under the previous assumptions:

$$\forall M \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall n \geq \Phi(M,m)\, \forall t \in [0,M]\, (\|T(t)x_n - x_n\| < 2^{-m})$$

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Example:

$\alpha = \sqrt{2}, \beta = 2, \lambda = 1/2$. Then $\Lambda = 2, N = 1, D = 2, f_\gamma(p) = 4p^2$.

Semigroups and Cauchy problems

- Semigroups play a central role in the study of Cauchy problems which are given by accretive operators in Banach spaces (Crandal-Liggett, Pazy, Barbu etc. since the 1970's).

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- Logical metatheorems covering the theory of nonlinear semigroups are due to Pischke 2023.

Pursuit-evasion games: Lion-Man

Let (X, d) be a uniquely geodesic space $D > 0$. $L_0, M_0 \in X$ starting points of the lion L and the man M. After i-steps, M moves to any point M_i s.t. $d(M_i, M_{i+1}) \leq D$ and L moves to L_{i+1} along the geodesic $[L_i, M_i]$ s.t. $d(L_i, L_{i+1}) = \min\{D, d(L_i, M_i)\}$.

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Proof mining yields (K./Lopéz-Acedo/Nicolae Pacific J. Math. 2021): simple **rate of convergence** and **vast generalization** of the class of permitted spaces X.

Proof Mining papers which appeared since 2023

- H. Cheval, L. Leuștean: Linear rates of asymptotic regularity for Halpern-type iterations. Mathematics of Computation (2024).
- H. Cheval, U. Kohlenbach, L. Leuștean: On modified Halpern and Tikhonov-Mann iterations. J. Optim. Theory Appl. (2023).
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- A. Freund, U. Kohlenbach: Bounds for a nonlinear ergodic theorem for Banach spaces. Ergodic Theory and Dynamical Systems (2023).
- B. Dinis, P. Pinto: Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces. SIAM J. Optim. (2023).
- R. Findling, U. Kohlenbach: Rates of convergence and metastability for Chidume's algorithm for the approximation of zeros of accretive operators in Banach spaces. Numerical Functional Analysis and Optimization (2024).

- U. Kohlenbach, N. Pischke: Proof theory and nonsmooth analysis. Phil. Trans. Royal Soc. A (2023).
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- M. Neri, T. Powell: computational study of a class of recursive inequalities. Journal of Logic and Analysis (2023).
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- P. Pinto: On the finitary content of Dykstra's cyclic projections algorithm. Zeitschrift für Analysis und ihre Anwendungen (2024).
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- P. Pinto, N. Pischke: On computational properties of Cauchy problems generated by accretive operators. Documenta Mathematica (2023).
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