The Curry-Howard correspondence journey

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Lecture 1

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correspondence

logic **computation**

proofs	–	terms
formulae	–	types
rules	–	reductions

Curry - Howard correspondence

INTUITIONISTIC LOGIC	COMPUTATION
Axiomatic system	Combinatory Logic
Natural Deduction	λ-calculus
Sequent system	sequent-like λ-calculi

coincidence

- 1950s Curry
- 1968 (1980) Howard - formulae-as-types
- 1970s Lambek - CCC Cartesian Closed Categories
- 1970s de Bruijn - AUTOMATH
- 1970s Martin-Löf - Type Theory

Journey

LOGIC	**COMPUTATION**
intuitionistic	λ-calculus combinatory logic
second-order	polymorphism
predicate	λ cube
classical	$\lambda\mu$-calculi
???	intersection types
communication	process calculi
??? federated (machine) learning	???

Roadmap Lecture 1

Logical systems

Lambda Calculus and Combinatory Logic

Simple (functional) types

Correspondence: Intuitionistic Logic and Computation

Roadmap

Logical systems

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Logical systems

LOGIC	
Axiomatic system	Hilbert style
Natural Deduction	Gentzen, Prawitz
Sequent system	Gentzen

Intuitionistic logic - axiomatic system

Brouwer (early 1900), Heyting (1960s)

$$\sigma \ ::= \ \alpha \mid (\sigma \to \sigma)$$

α ranges over a countable set of propositional variables

- **Axioms**

  (Ax 1) $\sigma \to (\tau \to \sigma)$

  (Ax 2) $(\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$

- **Inference rule**

  (*MP*)
  $$\frac{\sigma \to \tau \quad \sigma}{\tau}$$

- $\neg, \ \wedge, \ \vee, \ \leftrightarrow$ connectives are independent

- **BHK interpretation** - Brouwer, Heyting, Kolmogorov interpretation of logical connectives
  “math is a cognitive construct rather than a type of objective truth"
- Provable, theorems
  - $\vdash \sigma \to \sigma$, identity
  - $\vdash \sigma \to \neg\neg\sigma$, double negation introduction
  - $\vdash \neg\sigma \wedge \neg\tau \leftrightarrow \neg(\sigma \vee \tau)$, de Morgan law $(\frac{2}{4})$
  - $\vdash \neg\sigma \vee \neg\tau \to \neg(\sigma \wedge \tau)$, de Morgan law $(\frac{1}{4})$
- Not provable
  - $\nvdash ((\sigma \to \tau) \to \sigma) \to \sigma$, Peirce law
  - $\nvdash \sigma \vee \neg\sigma$, law of excluded middle (tertium non datur)
  - $\nvdash \neg\neg\sigma \to \sigma$, double negation elimination
  - $\nvdash \neg(\sigma \wedge \tau) \to \neg\sigma \vee \neg\tau)$, de Morgan law $(\frac{1}{4})$

Classical logic - axiomatic system

- **Axioms**

  (Ax 1) $\sigma \to (\tau \to \sigma)$

  (Ax 2) $(\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$

  (Ax 3) $(\neg\sigma \to \neg\tau) \to ((\neg\sigma \to \tau) \to \sigma)$

- **Inference rule**

  (*MP*) $$\frac{\sigma \to \tau \quad \sigma}{\tau}$$

- $\neg,\ \wedge,\ \vee,\ \leftrightarrow$ connectives are dependent
- Interdefinability of connectives: minimal bases for defining all connectives:
  - $\{\neg, \to\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$
  - $\{\downarrow\}$ Peirce arrow (NOR)
  - $\{\uparrow\}$ Sheffer stroke (NAND)

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Models of computation

- Alonzo Church:
  - theory of functions - formalisation of mathematics (inconsistent)
  - successful model for computable functions - **λ-calculus**
  - **simply typed λ-calculus**
- Haskell Curry:
  - elimination of variables in logic - Moses Schönfinkel (1921)
  - successful model for computable functions - **Combinatory logic**
  - **Combinatory logic with types**
- Alan Turing :
  - formalisation of the concepts of algorithm and computation
  - **Turing Machines**

λ-calculus - expressiveness

- Expressiveness - Effective computability (mid 1930s)
  - **(Curry)** Equivalence of λ-calculus and Combinatory Logic
  - **(Kleene)** Equivalence of λ-calculus and recursive functions
  - **(Turing)** Equivalence of λ-calculus and Turing machines

Combinatory logic

Curry, Schönfinkel (1921)

Syntax

$$M \ ::= \ x \mid \mathbf{S} \mid \mathbf{K} \mid \mathbf{I} \mid MM$$

Reduction rules

$$\begin{array}{lcl} \mathbf{I}M & \longrightarrow & M \\ \mathbf{K}MN & \longrightarrow & M \\ \mathbf{S}MNP & \longrightarrow & MP(NP) \end{array}$$

Example $\mathbf{SKK} = \mathbf{I}$

No bound variables ✓

λ-calculus - theory of functions

Syntax

$$M ::= x \mid c \mid (MM) \mid (\lambda x.M)$$

Reduction rules

α-reduction (renaming):

$$\lambda x.M \longrightarrow_\alpha \lambda y.M[x := y], \quad y \notin FV(M)$$

β-reduction (function evaluation):

$$(\lambda x.M)N \longrightarrow_\beta M[x := N]$$

Example

$(\lambda x.x)5 \rightarrow 5$ identity function

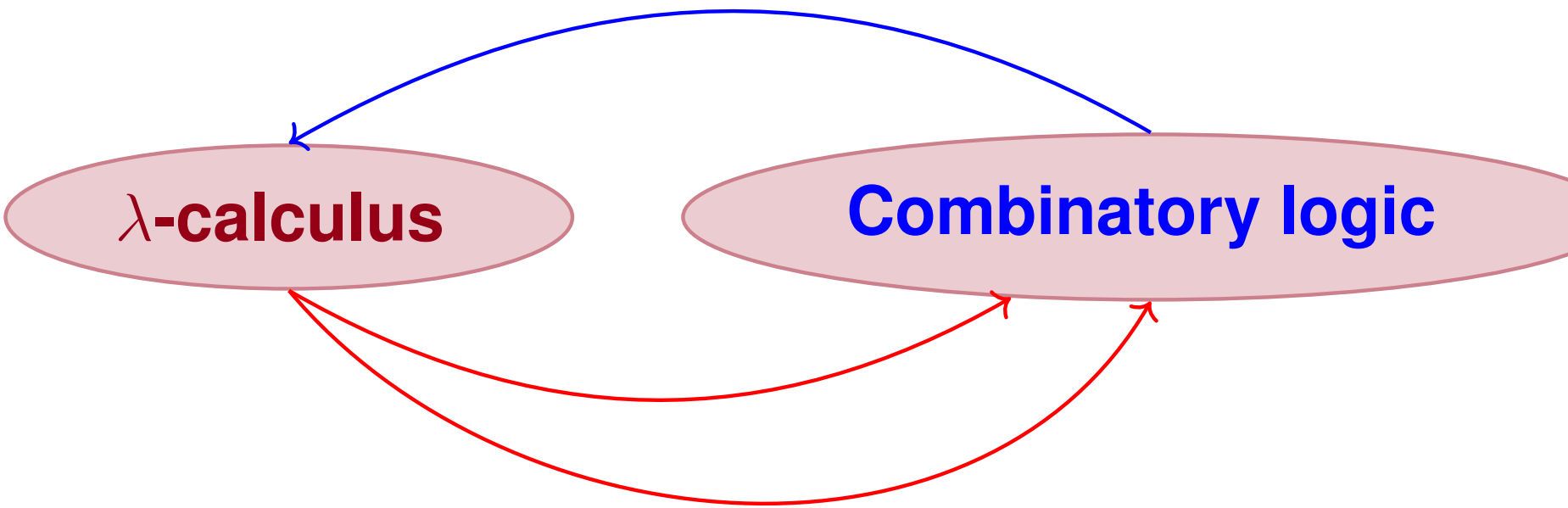
$(\lambda x.x^2 + 1)3 \rightarrow 3^2 + 1 = 10$

Equivalence of expressiveness

$$\mathbf{I} \equiv \lambda x.x$$

$$\mathbf{K} \equiv \lambda xy.x$$

$$\mathbf{S} \equiv \lambda xyz.xz(yz)$$



Running example

$xyzx$

$\lambda x.zx$

$\mathbf{I} \equiv \lambda x.x$	combinator **I**
$\mathbf{K} \equiv \lambda xy.x$	combinator **K**
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	combinator **S**
$\Delta \equiv \lambda x.xx$	selfapplication
$\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	fixed point combinator
$\Omega \equiv \Delta\Delta \equiv (\lambda x.xx)(\lambda x.xx)$	higher-order function

Properties

confluence	✓
normal forms	uniqueness (if exists)
normalisation	✗
strong normalisation	✗
fixed point theorem	✓
expressiveness	Turing machines, recursive functions

Properties - Confluence

Theorem (Church-Rosser theorem)

If $M \longrightarrow\!\!\!\!\!\rightarrow N$ and $M \longrightarrow\!\!\!\!\!\rightarrow P$, then there exists S such that $N \longrightarrow\!\!\!\!\!\rightarrow S$ and $P \longrightarrow\!\!\!\!\!\rightarrow S$

The proof is deep and involved.

Corollary

- *If $M \longrightarrow\!\!\!\!\!\rightarrow N$ and $M \longrightarrow\!\!\!\!\!\rightarrow P$, then $N = P$*
- *The order of the applied reductions is arbitrary and always leads to the same result*
- *Reductions can be executed in parallel (parallel computing)*

Proof.

Normal forms

- $N \in \Lambda$ is a normal form (NF) if there is no S such that $N \longrightarrow S$
- $P \in \Lambda$ is normalising (has a normal form) if $P \longrightarrow\!\!\!\!\!\rightarrow N$ and N is a normal form, then N is a NF of P
- $P \in \Lambda$ is strongly normalising (SN) if all reductions of P are finite

Notation: $\longrightarrow\!\!\!\!\!\rightarrow$ will denote $\longrightarrow\!\!\!\!\!\rightarrow_\beta \cup \longrightarrow_\alpha$

Theorem (uniqueness of NF)

Every lambda term has at most one normal form

Proof.

Exercise

$\mathbf{I} \equiv \lambda x.x$	normal form NF
$\mathbf{K} \equiv \lambda xy.x$	normal form NF
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	normal form NF
$\Delta \equiv \lambda x.xx$	normal form NF
$\mathbf{KI}(\mathbf{KII})$	strongly normalizing SN
$\Omega \equiv \Delta\Delta \equiv (\lambda x.xx)(\lambda x.xx)$	unsolvable
$\mathbf{KI}\Omega$	normalizing N
$\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$	head normalizing HN (solvable)

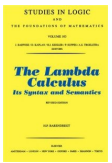
$\mathbf{KI}(\mathbf{KII}) \rightarrow \mathbf{KII} \rightarrow \mathbf{I}$

$\mathbf{KI}(\mathbf{KII}) \rightarrow \mathbf{I}$

$\Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \ldots$

$\mathbf{KI}\Omega \rightarrow \mathbf{I}$

References



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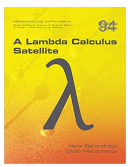

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Roadmap

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Lambda Calculus and Combinatory Logic

Simple (functional) types

Correspondence: Intuitionistic Logic and Computation

Simple (functional) types

Motivation

- “Disadvantages" of the untyped calculus:
  - infinite computation - there exist terms without a normal form
  - meaningless applications - it is allowed to create terms like **sin log**
- Types are syntactical objects that can be assigned to terms
  - Reasoning with types was present in the early work of Church and Curry on untyped calculi
- two typing paradigms:
  - *à la* Curry - implicit type assignment (*lambda calculus with types*)
  - *à la* Church - explicit type assignment (*typed lambda calculus*)

Simple (functional) types

- Language of types **Type**

$$\sigma \ ::= \ a \mid \sigma \to \sigma$$

- Type assignment

$M \in \Lambda(CL)$ $\qquad\qquad$ $\sigma \in$ **Type**

$$M : \sigma$$

$\lambda \rightarrow$ - the type system

à la Curry and à la Church

- Axiom

$$(Ax) \qquad \frac{}{\Gamma, x : \sigma \vdash x : \sigma}$$

- Rules

$$(\rightarrow_{elim}) \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau}\,(\rightarrow_{intr}) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.M : \sigma \rightarrow \tau}$$

à la Curry à la Church

Running example: types

M	Type
$\mathbf{I} \equiv \lambda x.x$	$\sigma \to \sigma$
$\mathbf{K} \equiv \lambda xy.x$	$\sigma \to (\tau \to \sigma)$
$\mathbf{S} \equiv \lambda xyz.xz(yz)$	$(\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$
$\Delta \equiv \lambda x.xx$	NO
$\mathbf{Y}$	NO
$\Omega \equiv \Delta\Delta$	NO

Properties

confluence	✓
normal forms	✓
normalisation	✓
strong normalisation	✓
fixed point theorem	✗
expressiveness	extended polynomials

Combinatory logic with simple types

- **Axioms**
  - **(Ax 1)** $\mathbf{K} : \sigma \to (\tau \to \sigma)$
  - **(Ax 2)** $\mathbf{S} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$
- **Inference rule**

$$(MP) \qquad \frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}$$

- **Y** and Ω are not typable

3 Type?

Type checking: given M and σ

$$(M : \sigma)?$$

Type inference (typability, type synthesis): given M

$$M :?$$

Type inhabitation (term, program synthesis) : given σ

$$? : \sigma$$

$\lambda \rightarrow$ 3 Type?

Theorem

In $\lambda \rightarrow$

- *Type checking* $((M : \sigma)?)$ *is decidable*
- *Type inference* $(M :?)$ *is decidable*
- *Type inhabitation* $(? : \sigma)$ *is decidable*

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Axioms and Combinators

Intuitionistic logic ✓

Classical logic ✗

- **Axioms**

(Ax 1) **K** $:\sigma \to (\tau \to \sigma)$

(Ax 2) **S** $:(\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$

(Ax 3) ✗ $: (\neg\sigma \to \neg\tau) \to ((\neg\sigma \to \tau) \to \sigma)\ (\neg\sigma \to \neg\tau) \to$

- **Inference rule**

(MP) $$\frac{M :\sigma \to \tau \quad N :\sigma}{MN :\tau}$$

Theorem (Curry Howard correspondence)

$\vdash \sigma(IL)$ *if and only if* $\vdash M : \sigma$

Theorem (Curry Howard correspondence)

$$\vdash \sigma \ (\mathit{PROP}) \quad \textit{if and only if} \quad \vdash M : \sigma$$

The correspondence

- holds for intuitionistic logic
- does not hold for classical logic

Natural deduction and λ

Intuitionistic logic (minimal) - Natural deduction, Gentzen 1930s

- Axiom

$$(Ax) \qquad \frac{}{\Gamma, \sigma \vdash \sigma}$$

- Rules

$$(\rightarrow_{elim}) \qquad \frac{\Gamma \vdash \sigma \rightarrow \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau}$$

$$(\rightarrow_{intr}) \qquad \frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \rightarrow \tau}$$

$\lambda \rightarrow$ and logic

Intuitionistic logic (minimal) - Natural deduction, Gentzen 1930s

- Axiom

$$(Ax) \qquad \frac{}{\Gamma, x : \sigma \vdash x : \sigma}$$

- Rules

$$(\rightarrow_{elim}) \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$(\rightarrow_{intr}) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau}$$

Curry-Howard correspondence

Intuitionistic logic vs computation

$$\vdash \sigma \quad \Leftrightarrow \quad \vdash M : \sigma$$

Theorem

A formula is provable in minimal intuitionistic logic
if and only if it is inhabited in $\lambda \rightarrow$

formulae	–as–	types
proofs	– as –	terms
proofs	–as–	programs
proof normalisation	–as–	term reduction
cut elimination	–as–	term reduction

- ✓ BHK - Brouwer, Heyting, Kolmogorov interpretation of logical connectives is formalized by the Curry-Howard correspondence
- $\mathcal{P}(A \to B)$ are the maps from $\mathcal{P}(A)$ into $\mathcal{P}(B)$
- Pierce's law not inhabited ?

Curry - Howard correspondence

INTUITIONISTIC LOGIC	COMPUTATION
Axiomatic system	Combinatory Logic
Natural Deduction	λ-calculus
Sequent system	sequent-like λ-calculi

References

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Lambda calculi with types

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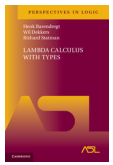

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